Whitening of Signals in Range to Improve Estimates of Polarimetric Variables

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ABSTRACT

A method to reduce errors in estimates of polarimetric variables beyond those achievable with standard estimators is suggested. It consists of oversampling echo signals in range, applying linear transformations to decorrelate these samples, processing in time the sequences at fixed range locations to obtain various second-order moments, averaging in range these moments, and, finally, combining them into polarimetric variables. The polarimetric variables considered are differential reflectivity, differential phase, and the copolar correlation coefficient between the horizontally and vertically polarized echoes. Simulations and analytical formulas confirm a reduction in variance proportional to the number of samples within the pulse compared to standard processing of signals behind a matched filter. This reduction is possible, however, if the signal-to-noise ratios (SNRs) are larger than a critical value. Plots of the critical SNRs for various estimates as functions of Doppler spectrum width and other parameters are provided.

1. Introduction

Evidence accumulated by researchers indicates that the operational application of radar polarimetry is a distinct possibility, and plans exist to incorporate this capability into the U.S. national radar network. One remaining practical issue concerns the rather stringent requirements on the errors in polarimetric variables. This needs to be addressed if the full potential of this technology is to be realized. Although the initial implementation of processing schemes is far from being decided, it is clear that with the current volume-update rates and signal-processing algorithms, the polarimetric variables will have errors larger than those expected by some researchers. It is submitted that, even with larger errors, addition of polarimetry will benefit the operational community, because the errors of the other spectral moments will remain the same and the new polarimetric variables will add additional information useful for precipitation measurements, identification of precipitation type, and data quality control. Hence, it seems that one cannot but gain with the addition of polarimetry.

The purpose of this paper is to suggest a processing scheme that can improve the estimates of polarimetric variables to meet the stringent error requirements so that their usefulness is brought in line with that of the other spectral moments. Weather radar signal processing has matured over the years so that significant breakthroughs are less likely, although improvements can still be made. Some improvements are becoming practical because today's faster processors allow the real-time implementation of complex algorithms. Thus, the proposed scheme is pertinent.

Estimates of spectral moments in weather radars are made from a number of returned signals, usually tens. Because these signals are correlated, radars dwell long enough to obtain a sufficient (equivalent) number of independent samples. As is well known from statistics, the larger the number of independent samples, the lower the error of the estimate derived from those samples. Therefore, when error requirements are stringent, the ability to obtain a large number of independent samples becomes critical. Indeed, the main reason for averaging reflectivity in range on the Weather Surveillance Radar-1988 Doppler (WSR-88D) (over a 1-km interval, or four samples) is to increase the equivalent number of independent samples. Analogously, the current implementation of polarimetric variable estimators[1] on research

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[1] The analysis in this paper is for simultaneous transmission and reception of horizontally and vertically polarized signals (e.g., Doviak et al. 2000; Scott et al. 2001). Nonetheless, the same principle is applicable to alternate (switched) transmissions and receptions.

radars uses a simple method of averaging samples in range at the expense of degradation in range resolution. Because simple averaging does not yield the best performance if the observations are correlated, it is often the practice to slow down the antenna so that there are more samples in the dwell time and the desired statistical error of estimates is attained. However, this is not a desirable solution because an increase in update times reduces the ability to detect and study fast-evolving meteorological phenomena.

To maintain the current update times, Schulz and Kostinski (1997) first suggested that knowledge of the correlation structure of weather signals could be used to formulate polarimetric variable estimators that attain the Cramer–Rao lower bound (i.e., the best theoretical performance) based on a whitening transformation at high signal-to-noise ratios (SNRs). A principal obstacle to the application of the whitening transformation on the time series is the need to known (or estimate) the correlation coefficient of the signal and its Gaussian nature. Schulz and Kostinski suggest ways to overcome these difficulties and call for an experimental study to verify the technique (Koivunen and Kostinski 1999, 2000). Although the concept of whitening is mature, it was not applied to radar remote sensing until the last decade. In a rather short but important study, Dias and Leitão (1993) derived an iterative technique to obtain maximum likelihood (ML) estimates of spectral moments. They consider both the time and space variables and assume a known correlation in range. Implicit in their solution is whitening of the signals in range, and thus it is likely the earliest application of whitening in the context of signal processing for weather radars. Alternatively, pulse compression can be applied to increase the equivalent number of independent samples by averaging high-resolution estimates in range (Mudukutore et al. 1998). However, most ground-based weather surveillance radar do not use pulse compression because of the required larger transmission bandwidths.

Torres and Zrnić (2003) suggested application of the whitening transformation to oversampled data along range time for estimating Doppler spectral moments. This paper is a follow-up on that suggestion; it applies the whitening transformation to the estimation of polarimetric variables. Throughout this study we deal with an ideal system and a known range-time correlation coefficient, which derives from assumptions of frozen scatterers and uniform reflectivity in the resolution volume (these assumptions are justified in section 2). For a discussion on the effects of reflectivity gradients the reader is referred to Torres (2001) and Torres and Zrnić (2003). Although a rectangular pulse and infinite bandwidth receiver are not realistic, they closely approximate the characteristics of operational systems and serve to illustrate the potential of the method. For example, an intermediate frequency (IF) bandwidth of about 10 times the reciprocal of the pulse width will be enough to allow for oversampling and still have the noise remain white compared to the weather signal bandwidth. In what follows, we review the whitening transformation on oversampled range data from which polarimetric variables estimators are derived. Then, we discuss the results obtained from theory and simulations. Derivations of the variance of estimates and details of the simulation procedures are in the appendixes.

2. Oversampling in range and the whitening transformation

Oversampling in range entails acquiring polarimetric time series data at increased rates so that there are L complex samples at horizontal (V_H) and vertical (V_V) polarization during the pulse duration τ. This is referred to as oversampling by a factor of L. Samples obtained in this way are correlated along range time, and their correlation is needed to determine a whitening (or decorrelation) transformation.

Generally, the autocorrelation of the signal $V(r, t)$ is a function of the transmitted pulse shape, the receiver's impulse response, the distribution of scatterers in the resolution volume, and their relative motion, that is, the random velocity field $v(r, y)$, where y is the transverse to the beam distance. However, in applications pertinent to weather surveillance radars, the contribution of short-term temporal correlation due to the random velocity field can be neglected. This is justified as follows. The contribution of scatterers to the oversampled signals in range occurs in an extremely short time (of the order of 1 μs) during which the short pulse ($\sim$1.6 μs for the WSR-88D) traverses its length. During this short time the cause of decorrelation is primarily the progressive excitation of new scatterers in range as some scatterers are left behind; there is an imperceptible contribution from scatterer motion as the following example illustrates. Take a 1-μs pulse (typical for weather radar) and a very large velocity distribution of scatterers (either random or deterministic motion); let this distribution extend to 100 m s^{-1}. It can be seen that the largest relative displacement between a stationary and the fastest scatterer during 1 μs is 0.1 mm. This is 250 times shorter than a quarter of the wavelength for a 10-cm radar like the WSR-88D. Clearly, relative motion of scatterers in that period has negligible effect on the signal, and the main decorrelating mechanism comes from the pulse propagation through a random distribution of scatterers. In summary, the velocity field can be ignored when computing the range-time correlation of oversampled data (frozen scatterers).

The assumption of uniform reflectivity (uniform distribution of scatterers), which was mentioned in the introduction, is needed to ensure that the correlation function of oversampled signals along range time can be computed a priori. That is, assuming that the range of depth $c\tau$ (where c is the velocity of light) is uniformly filled with (frozen) scatterers, which is a common occurrence for relatively short pulses, the correlation coefficient of these range samples $\rho_V^{(R)}$ is determined by the pulse shape and the receiver filter impulse response as (Torres 2001)

$$\rho_V^{(R)}(l) = \frac{R_V^{(R)}(l)}{R_V^{(R)}(0)} = \frac{p_m(l) \star p_m^*(-l)}{\sum_{l'=0}^{L-1} p_m^2(l')}, \qquad (1)$$

where the superscript (R) indicates range-time correlations, and the symbols $\star$ and $*$ stand for convolution and complex conjugation, respectively. In addition, $p_m(l) = p(l) \star h(l)$, where $p(l)$ is the transmitted pulse shape, and $h(l)$ is the impulse response of the receiver filter. Equation (1) is valid for either polarization and therefore uses subscript V to indicate a generic radar return signal (V_H or V_V) of arbitrary polarization. In fact, under the assumptions stated above, the range-time correlation depends solely on parameters that are known or can be precisely measured. That is a distinct advantage of the proposed whitening scheme.

The whitening transformation is obtained from the known range-time correlation coefficient $\rho_V^{(R)}$ by decomposing the associated Hermitian Toeplitz correlation coefficient matrix $\mathbf{C}_V^{(R)}$ as

$$\mathbf{C}_V^{(R)} = \mathbf{H}^*\mathbf{H}^{\mathrm{T}}, \qquad (2)$$

where the superscript T indicates matrix transpose. Any $\mathbf{H}$ that satisfies (2) is called a square root of $\mathbf{C}_V^{(R)}$ and is the inverse of a whitening transformation matrix $\mathbf{W}$, which if applied to the range samples $V(0, n)$, $V(1, n)$, $\ldots$, $V(L-1, n)$ (sample time n is fixed) produces L uncorrelated random variables. The uncorrelated (whitened) sequence, denoted by $X(l, n)$, is the sequence of range samples spaced by the pulse repetition time obtained as

$$X(l, n) = \sum_{j=0}^{L-1} w_{l,j} V(j, n) \quad \text{for } l = 0, 1, \ldots, L-1, \qquad (3)$$

where $w_{l,j}$ are the entries of $\mathbf{W} = \mathbf{H}^{-1}$. Equation (3) applies to both the horizontal and vertical channels, producing X_H and X_V from V_H and V_V, respectively.

Throughout this paper a whitening transformation that is derived only from the weather signal correlation properties is considered; the additive noise is not included in its derivation. Still, the whitening transformation affects both signal and noise evenly. While the signal is perfectly whitened, the noise, which was white to begin with, becomes colored and is amplified. It can be shown (Torres and Zrnić 2003) that for a correlation matrix corresponding to an ideal transmitter/receiver system (i.e., a system with a rectangular transmitted pulse and radar bandwidth much larger than the reciprocal of pulse width) the so-called *noise-enhancement factor* (NEF) is

$$\mathrm{NEF} = \frac{N_w}{N} = \frac{L^2}{L+1}, \qquad (4)$$

where N is the receiver noise power and N_w is the noise power after the whitening transformation. Note that an extra L factor should be added if comparing the noise power with the classical processing. To effectively oversample by a factor of L, an L-times-larger bandwidth than the reciprocal of the pulse width is needed; this increases the noise power by the same factor. Under these considerations, the effective noise increase over the matched filter case (for an ideal system) becomes $L^3(L + 1)^{-1}$.

The trade-off between noise enhancement (radar sensitivity) and variance reduction makes the whitening transformation useful in cases of relatively large SNR. For weather surveillance radars, the SNR of signals from storms is large and the effects of noise when using the whitening transformation are negligible. For example, a 3 mm h^{-1} rain in the WSR-88D produces an SNR of 37 dB at 50 km. Clearly, for measuring light rain, the noise enhancement by whitening would not be a problem to the full of 230 km of required coverage. The threshold for display is set at a relatively small SNR ($<$3.5 dB) mainly to observe snow, which has typically smaller reflectivity. This SNR falls in the range where noise enhancement dominates and may preclude the use of whitening.

An alternative to the whitening transformation that reduces the noise enhancement problem is based on the minimum mean-square error (mmse) criterion. Such transformation has the desired properties but needs a priori knowledge of the SNR at every range location (Ebbini et al. 1993). Alternatively, the same problem can be studied in terms of the eigenvalues of the range-time correlation matrix. The ability to limit the gain of the whitening transformation to reduce the noise enhancement effect arises from the relation between the eigenvalues of a correlation matrix and the corresponding power spectral density. That is, the range spanned by the power spectral density matches closely the range of eigenvalues (Johnson and Dudgeon 1993). Accordingly, by limiting the span of eigenvalues, it is possible to place a bound on the gain of the transformation (Torres 2001). The analysis of these and other suboptimal techniques is a subject for further study.

3. Polarimetric variable estimators

a. Differential reflectivity estimator

The differential reflectivity Z_{DR} is the ratio of reflected horizontal and vertical power returns. The capability of dual-polarized radars to estimate rainfall rate with better accuracy via differential reflectivity measurement has been well established (Aydin et al. 1990). In addition, Z_{DR} makes identification of hail possible (Zrnić and Ryzhkov 1999). However, one of the problems with Z_{DR} measurement has been its relatively long acquisition time because accurate rainfall-rate estimation requires Z_{DR} fractional errors less than 0.2 dB; hence, with the current implementation more samples, and thus a slower antenna rotation rate, are necessary.

The whitening-transformation-based (WTB) differential reflectivity (in linear scale) estimator for oversampled signals in noise is given by

$$\hat{Z}_{\underset{(\mathrm{WTB})}{\mathrm{DR}}} = \hat{S}_{X_H}/\hat{S}_{X_V}, \tag{5}$$

where the signal powers in the horizontal and vertical channels are estimated as

$$\hat{S}_{X_H} = \frac{1}{LM}\sum_{l=0}^{L-1}\sum_{m=0}^{M-1}|X_H(l, m)|^2 - N(\mathrm{NEF}), \tag{6}$$

$$\hat{S}_{X_V} = \frac{1}{LM}\sum_{l=0}^{L-1}\sum_{m=0}^{M-1}|X_V(l, m)|^2 - N(\mathrm{NEF}). \tag{7}$$

In the previous equations L is the oversampling factor, M is the number of pulses, N is the receiver noise power (the same for both the horizontal and vertical channels), NEF is the noise-enhancement factor as defined in the previous section, and $X_H(l, n)$ and $X_V(l, n)$ are the whitened oversampled weather signals for the horizontal and vertical channels, respectively.

Using the results in appendix A for an ideal system, the normalized standard deviation of differential reflectivity estimates is obtained from (A41) as

$$\frac{\mathrm{SD}\{\hat{Z}_{\underset{(\mathrm{WTB})}{\mathrm{DR}}}\}}{Z_{\mathrm{DR}}} = \frac{1}{\sqrt{M}}\left[\left(\frac{1-\rho_{\mathrm{HV}}^2}{\sigma_{vn}\sqrt{\pi}}\right)\frac{1}{L} + 2(1+Z_{\mathrm{DR}})\frac{L}{L+1}\left(\frac{N}{S_H}\right) + (1+Z_{\mathrm{DR}}^2)\frac{L(3L^2+2L-3)}{2(L+1)^2}\left(\frac{N}{S_H}\right)^2\right]^{1/2}, \tag{8}$$

where σ_{vn}, the normalized spectrum width, is defined as $\sigma_v/2v_a$; σ_v is the spectrum width; v_a is the maximum unambiguous velocity; and ρ_{HV} is the magnitude of the zero-lag, cross-correlation coefficient between horizontally and vertically polarized returns.

b. Differential phase estimator

The difference between the phases of horizontally and vertically polarized returns defines this parameter. Backscattering, propagation, and system effects are included in the differential phase ϕ_{DP}. Specific differential phase K_{DP} defined as the range derivative of ϕ_{DP} is the pertinent polarimetric variable for measuring rainfall (Sachidananda and Zrnić 1986). Although K_{DP} has some distinct advantages over other polarimetric variables (Zrnić and Ryzhkov 1996), its full utility and the best way to incorporate it into rainfall estimators is still an active area of research. It is known, however, that the standard error of K_{DP} is almost independent of rain rate, and this adversely affects the measurement of light rainfall. The standard error in K_{DP} is linearly related to the standard error in ϕ_{DP}; hence, reduction of the SD (ϕ_{DP}) can significantly improve measurement of rainfall. It is desirable to keep this error below 1°. Differential phase obtained with the necessary accuracy may also prove to be very useful in hydrometeor type identification (Zrnić and Ryzhkov 1999).

The WTB differential phase estimator for oversampled signals is given by

$$\hat{\phi}_{\underset{(\mathrm{WTB})}{\mathrm{DP}}} = \arg\{\hat{R}^{(T)}_{X_H X_V}(0)\}, \tag{9}$$

where the lag-zero sample-time cross-correlation function between the oversampled whitened signals $X_H(l, n)$ and $X_V(l, n)$ is estimated as

$$\hat{R}^{(T)}_{X_H X_V}(0) = \frac{1}{LM}\sum_{l=0}^{L-1}\sum_{m=0}^{M-1} X_V^*(l, m)X_H(l, m), \tag{10}$$

and where $X_H(l, n)$ and $X_V(l, n)$ are the whitened oversampled weather signals obtained from (3). Superscript (T) stands for sample-time correlation. From Eq. (A42) in appendix A, the standard deviation (in degrees) of differential phase estimates for the ideal case is

$$\mathrm{SD}\{\hat{\phi}_{\underset{(\mathrm{WTB})}{\mathrm{DP}}}\} = \frac{90}{\pi\sqrt{2M}}\left[\left(\frac{\rho_{\mathrm{HV}}^{-2}-1}{2\sigma_{vn}\sqrt{\pi}}\right)\frac{1}{L} + \left(\frac{1+Z_{\mathrm{DR}}}{\rho_{\mathrm{HV}}^2}\right)\frac{L}{L+1}\left(\frac{N}{S_H}\right) + \left(\frac{Z_{\mathrm{DR}}}{\rho_{\mathrm{HV}}^2}\right)\frac{L(3L^2+2L-3)}{2(L+1)^2}\left(\frac{N}{S_H}\right)^2\right]^{1/2}. \tag{11}$$

c. Magnitude of zero-lag cross-correlation coefficient estimator

The magnitude of the cross-correlation between the reflected horizontal and vertical voltage returns at lag-zero ρ_{HV} is a good indicator of regions where there is a mixture of precipitation types, such as rain and snow. The magnitude of the zero-lag cross-correlation coefficient, loosely referred to in the literature as simply the "cross-correlation coefficient," depends on the shape, oscillation, wobbling, and canting angle distribution of hydrometeors (Sachidananda and Zrnić 1985). This polarimetric variable has been recently investigated for application to hail sizing, improving polarization estimates of rainfall, and detection of melting level in both convective and stratiform precipitation (Liu et al. 1994). As with the other polarimetric variables, the performance of these algorithms highly depends on the ability to obtain low-error estimates of ρ_{HV}.

The WTB cross-correlation coefficient estimator for oversampled signals is given by

$$\hat{\rho}_{\underset{(\mathrm{WTB})}{\mathrm{HV}}} = \frac{|\hat{R}^{(T)}_{X_H X_V}(0)|}{\sqrt{\hat{S}_{X_H}\hat{S}_{X_V}}}, \tag{12}$$

where $\hat{S}_{X_H}$, $\hat{S}_{X_V}$, and $\tilde{R}^{\mathrm{T}}_{X_H X_V}(0)$ are given in Eqs. (6), (7), and (10), respectively. From (A43), the standard deviation of cross-correlation coefficient estimates for the ideal case is

$$\mathrm{SD}\{\hat{\rho}_{\mathrm{HV}}\}_{(\mathrm{WTB})} = \frac{1}{\sqrt{M}}\left\{\left(\frac{1 - 2\rho_{\mathrm{HV}}^2 + \rho_{\mathrm{HV}}^4}{4\sigma_{vn}\sqrt{\pi}}\right)\frac{1}{L} + \left[\frac{(1 - \rho_{\mathrm{HV}}^2)(1 + Z_{\mathrm{DR}})}{2}\right]\frac{L}{L + 1}\left(\frac{N}{S_H}\right) + \left(\frac{Z_{\mathrm{DR}}}{\rho_{\mathrm{HV}}^2}\right)\frac{L(3L^2 + 2L - 3)}{2(L + 1)^2}\left(\frac{N}{S_H}\right)^2\right\}^{1/2}. \quad (13)$$

d. Results

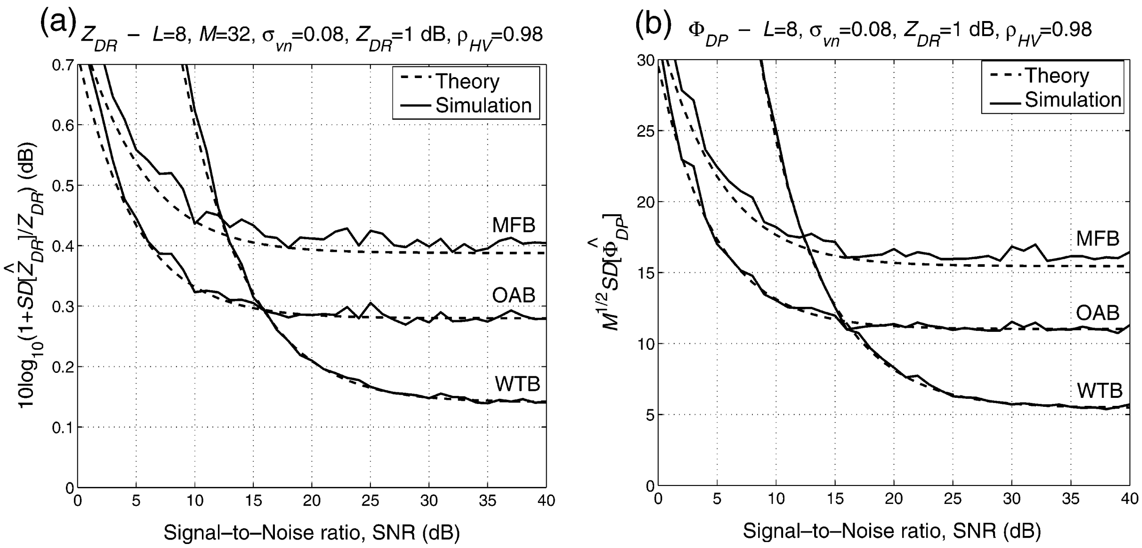
The performance of WTB polarimetric estimators is assessed with respect to the classical matched-filter-based (MFB) estimators and the estimators obtained from oversampled data and regular averaging. MFB estimators are obtained from oversampled data using coherent range averaging, that is, the type of conventional processing that uses a digital matched filter at the receiver's front end (averaging in range is performed at the I and Q component level). Alternatively, oversampling-and-average-based (OAB) estimators operate on oversampled data but using incoherent averaging; that is, averaging in range is performed at the correlation level. Throughout this work we stress the comparison of WTB estimates, the proposed implementation, with MFB estimates, the current implementation in the WSR-88D. OAB estimators achieve a modest improvement but are computationally simpler and not susceptible to reflectivity gradients or noise-enhancement effects. Thus, their performance is included to supplement our analysis, and the reader is referred to Torres (2001) for more details.

Figure 1 shows the normalized standard errors of WTB, MFB, and OAB (a) differential reflectivity, (b) differential phase, and (c) cross-correlation coefficient estimators as a function of the SNR for the ideal case, a normalized spectrum width of 0.08, and true Z_{DR} of 1 dB and ρ_{HV} of 0.98, which are representative of pure rain (Straka et al. 2000). The oversampling factor $L = 8$ is a realistic value that can be achieved on weather surveillance radar (Ivic et al. 2003). When compared to MFB (or OAB) estimates, WTB estimates exhibit a superior performance for large SNR. However, it is evident from these plots that the performance of the WTB estimator worsens as the SNR decreases due to the noise-enhancement effect inherent to the whitening transformation. The performance of OAB estimators is not optimum in terms of variance reduction because the average is done on correlated variables. However, since there is no noise enhancement, these estimates exhibit lower errors than MFB estimates for a broader range of SNR. The variance reduction factor (VRF) for the three WTB estimators in relation to the conventional matched filter approach can be computed as the ratio of the variance of MFB estimates to the variance of WTB estimates. For the ideal receiver case and a large SNR, the VRF is equal to the oversampling factor L [cf. (A41)–(A43) with (A47)–(A49), respectively]. Note that these results are obtained under the assumption of uniform reflectivity and frozen scatterers, and that theoretical predictions are in good agreement with simulation results.

The choice of parameters in Fig. 1 corresponds to values typical for weather surveillance radars that operate at the 10-cm wavelength. For example, a pulse repetition time of 1 ms produces a v_a of 25 m s^{-1}. With this v_a the σ_v (from Fig. 1) becomes 4 m s^{-1}, which is the value used to specify errors in spectral moments on the WSR-88D. The results in Fig. 1 can be used to obtain actual errors under particular conditions. For example, consider a number of samples $M = 32$, which produces a dwell time of 32 ms, which is about as short as it can be on the WSR-88D; hence, the computations are for the worst case. Also, take an SNR of 30 dB, which corresponds to the point where the curves begin to flatten, that is, the point at which WTB estimators approach the theoretical noise-free performance. Under the previous condition and assuming a Z_{DR} of 1 dB, the standard error of Z_{DR} estimates is 0.044 for WTB estimates and 0.123 for the MTB counterparts. Similarly, the actual errors of ϕ_{DP} estimates are 1.035° and 2.85° for WTB and MTB estimates, respectively. The errors of ρ_{HV} estimates are 3.6×10^{-3} for WTB and 10.2×10^{-3} for MTB. Note that in three cases, the ratio of MTB to WTB errors approximates $L^{1/2} = 2.828$, as expected.

Figure 2 shows the bias of WTB, MFB, and OAB (a) differential reflectivity, (b) differential phase, and (c) cross-correlation coefficient estimators as a function of the signal-to-noise ratio under the same conditions as in Fig. 1. Although all estimators are unbiased for large SNR, WTB Z_{DR} and ρ_{HV} estimators exhibit a bias at low SNR due to the noise-enhancement effect. Note that ϕ_{DP}, being a phase estimator, is more immune to the effects of noise.

It is of practical significance to determine the value of SNR for which the variance of estimates obtained from whitened samples is equal to the variance of estimates in the matched-filter implementation. This is because WTB estimators should be utilized only for SNRs greater than or equal to the crossover point SNR$_c$. By definition, SNR$_c$ is found by equating the VRF to one. SNR$_c$ is plotted in Fig. 3 versus the normalized spectrum width for (a) differential reflectivity, (b) differential phase, and (c) cross-correlation coefficient estimates. For completeness, the SNR$_c$ with respect to OAB estimates is also shown. These plots for different values of Z_{DR} were obtained directly from the theoretical results derived in appendix A and



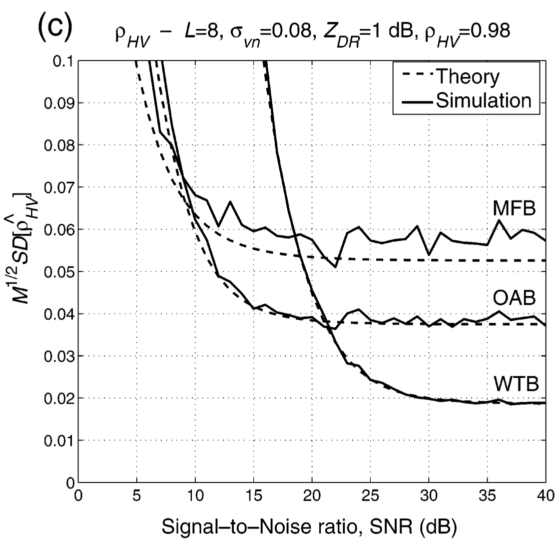


FIG. 1. Standard error of (a) differential reflectivity, (b) differential phase, and (c) cross-correlation coefficient estimates vs the SNR for the ideal case. The three sets of curves in each figure correspond to WTB, MFB, and OAB estimators, respectively. Solid lines show the results from simulations (averaging 1000 realizations) and dashed lines the theoretical predictions.

verified through simulation studies (see appendix B). Differential reflectivity values between 1 and 4 dB are representative of pure rain (Straka et al. 2000).

4. Discussion

In section 3 the application of the whitening transformation to the estimation of polarimetric variables is discussed. Estimators operating on whitened signals were termed WTB estimators, and they exhibit reduced standard errors with respect to MFB (or OAB) counterparts if the SNR is relatively large.

A performance comparison of all WTB with MFB polarimetric estimators for the ideal case showed that for all the variables the VRF for large SNR is L^{-1}, where L is the oversampling factor. That is, approximately L times fewer samples are needed for WTB estimators to keep the same errors as the ones obtained without the aid of the whitening transformation. Conversely, for low SNR the performance of all WTB estimators deteriorates as the noise-enhancing effect discussed in section 2 becomes important. In such cases, the estimates on match-filtered data result in better performance; the rule for selecting the best estimate is given by

$$\hat{\theta} = \begin{cases} \hat{\theta}_{(\mathrm{MFB})} & \text{if SNR} \leq \mathrm{SNR}_c \\ \hat{\theta}_{(\mathrm{WTB})} & \text{if SNR} > \mathrm{SNR}_c. \end{cases} \tag{14}$$

In the previous equation, θ is any of the variables discussed in this paper, namely Z_{DR}, ϕ_{DP}, and ρ_{HV}, and SNR_c is the crossover SNR (different for each estimator) defined as the SNR that conduces to a variance reduction factor of one. Theoretical expressions for the variance of estimates as derived in appendix A become useful if one needs to identify the value of

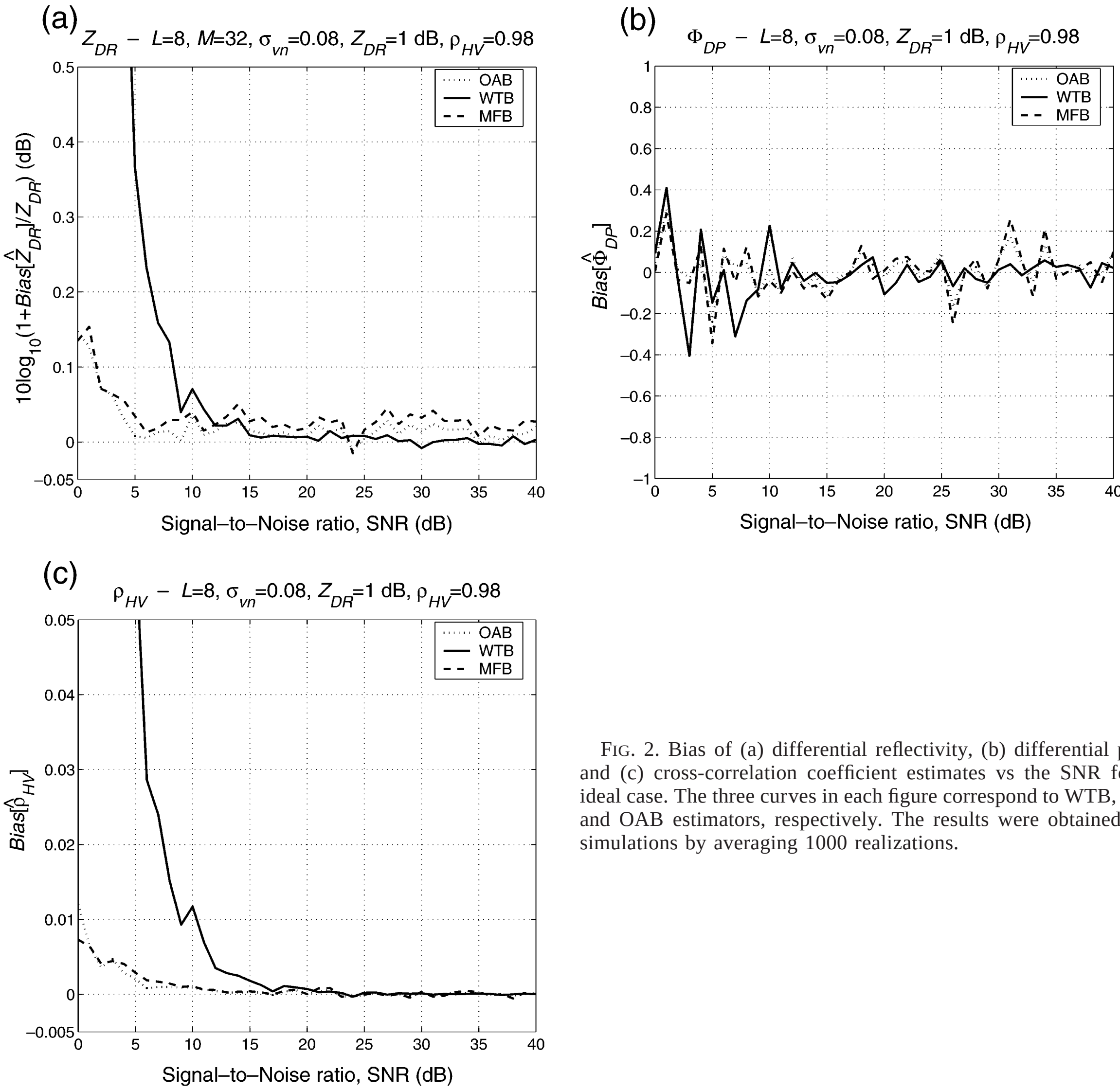


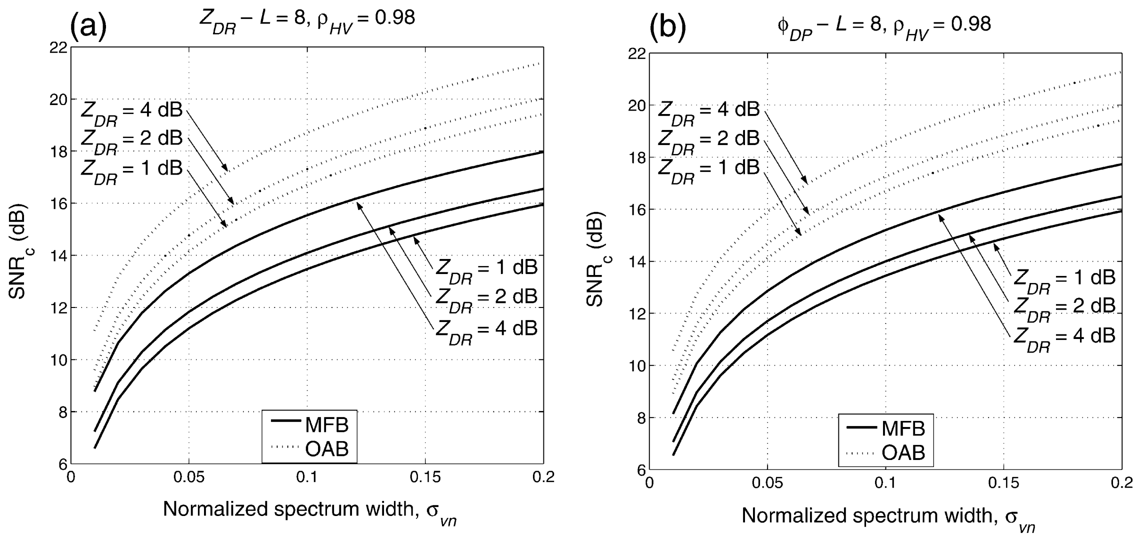
FIG. 2. Bias of (a) differential reflectivity, (b) differential phase, and (c) cross-correlation coefficient estimates vs the SNR for the ideal case. The three curves in each figure correspond to WTB, MFB, and OAB estimators, respectively. The results were obtained from simulations by averaging 1000 realizations.

SNR_c for a given variable under specific conditions without the need for simulations (see Fig. 3). Under appropriate limiting conditions, the theoretical variances agree with the ones available in the literature [Eq. (7) of Sachidananda and Zrnić (1985) for Z_{DR} and Eq. (A18) of Ryzhkov and Zrnić (1998) for ϕ_{DP}]. Finally, it is interesting to observe that the SNR_c for the differential reflectivity and differential phase are very similar (Figs. 3a and 3b), while the one for the cross-correlation coefficient (Fig. 3c) is larger, making $\hat{\rho}_{HV}$ a more sensitive estimator in terms of the whitening transformation.

The variance reduction obtained with WTB estimators is of considerable importance for the polarimetric variables. Unlike errors in the spectral moments, errors in polarimetric variables at the current antenna rotation rates do not always meet the required accuracy. Consequently, the use of WTB estimators for the polarimetric variables can reduce errors to acceptable levels without sacrificing the current update times (i.e., without slowing down the antenna rotational speed).

5. Conclusions

A method for the estimation of polarimetric variables on pulsed weather radars was presented. The scheme operates on oversampled echoes in range; that is, samples of in-phase and quadrature-phase components are taken at a rate L times larger than the reciprocal of the transmitted pulse length. Each set of L correlated samples is next transformed into a set of L decorrelated samples using a whitening transformation. Powers and correlations are estimated in the usual way along sample time, resulting in L values for each of these estimated quantities. The L values for each estimate are subsequently averaged, and the



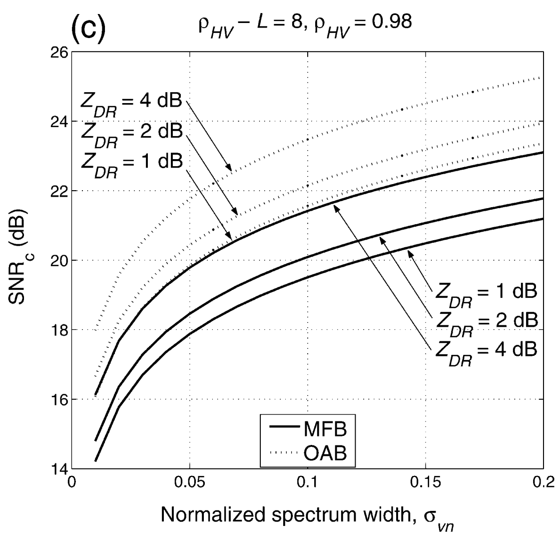


FIG. 3. Crossover signal-to-noise ratio (SNRc) vs the normalized spectrum width for (a) differential reflectivity, (b) differential phase, and (c) cross-correlation coefficient estimators in an ideal system. Solid lines represent the SNR at which the errors of WTB estimates equal the errors of MFB estimates. Dashed lines represent the SNR at which the errors of WTB estimates equal the errors of OAB estimates. WTB estimates are accepted if SNR $>$ SNR$_c$; otherwise, traditional estimates are preferred.

classical algorithms can then be used to compute the polarimetric variables. Because powers and correlations are derived from a set of decorrelated samples, the variance of polarimetric variable estimates decreases significantly.

Whitening-transformation-based (WTB) estimators of polarimetric variables were introduced, and their performance was compared to that of classical estimators that use a digital matched filter. The variance reduction (with respect to matched-filter-based estimators) achieved by WTB estimators under ideal conditions asymptotically tends to the inverse of the oversampling factor, L^{-1}, for signal-to-noise ratios (SNRs) of about 20 dB or more. For low SNRs there is a crossover point (SNR$_c$) for the variances of WTB and classical estimators. Analytical expressions that allow the computation of SNR$_c$ for any variable and different conditions were derived. For SNRs larger than the SNR$_c$, WTB estimates are preferred over classical estimates. Below the SNR$_c$, the noise-enhancement effect becomes more important and classical estimates are favored. Further, under conditions that violate the assumptions stated in section 2, WTB estimators will not perform as predicted.

The application of this technique is possible for the following reasons.

- The correlation of samples in range is known exactly if the resolution volume is uniformly filled with scatterers (suboptimum estimate variance reduction is also possible in resolution volumes with reflectivity gradients).
- The receiver bandwidth is large compared to the reciprocal of the pulse length.
- For most weather phenomena of interest, the SNR is relatively high (above the SNR$_c$), so the increase of noise power is not critical and the method works well.

Realistic simulations (see appendix B) were performed using known statistical properties of signals reflected by passive scatterers in fluids; known properties of the probing pulse and receiver filter are used to reconstruct a composite signal from distributed scatterers illuminated by the pulse. This work confirms that WTB polarimetric estimators are indeed viable candidates for future application on the WSR-88D radar network.

Summarizing, WTB estimators of polarimetric variables allow the currently accepted speeds of volume coverage by weather radar to be maintained, so that hazardous features can be detected in a timely fashion. This is done with minimal sacrifice in range resolution and without broadening the transmission bandwidth of the radar. Thus, on the future polarimetric WSR-88D the existing spectral moment estimators may remain the same, and the whitening transformation could be applied for estimating the additional polarimetric variables. That way, there would be no sacrifice of the present capabilities regardless of whether the noise is present or not. The acquisition of more accurate polarimetric variables without compromising any of the critical operation requirements would add value by leading to better products with acceptable statistical errors. Ultimately, these would improve rainfall estimates, identification of precipitation types, and data quality control.

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APPENDIX A

Derivation of Estimators' Variances

Estimators of the polarimetric variables $\hat{Z}_{\mathrm{DR}}$, $\hat{\phi}_{\mathrm{DP}}$, and $\hat{\rho}_{\mathrm{HV}}$ are obtained from estimates of total powers for the horizontal and vertical channels $\hat{P}_H$ and $\hat{P}_V$ and the sample-time (T) lag-zero cross-correlation function $\hat{R}^{\mathrm{T}}_{\mathrm{HV}}(0)$ as

$$\hat{Z}_{\mathrm{DR}} = \frac{\hat{P}_H - N}{\hat{P}_V - N}, \tag{A1}$$

$$\hat{\phi}_{\mathrm{DP}} = \arg\{\hat{R}^{(T)}_{\mathrm{HV}}(0)\}, \tag{A2}$$

$$\hat{\rho}_{HV} = \frac{|\hat{R}^{(T)}_{\mathrm{HV}}(0)|}{\sqrt{(\hat{P}_H - N)(\hat{P}_V - N)}}, \tag{A3}$$

where N is the receiver noise power in either channel.

Consider first the case of WTB estimators. Estimates of power and lag-zero cross-correlation are given by

$$\hat{P}_{X_H} = \frac{1}{LM}\sum_{l=0}^{L-1}\sum_{m=0}^{M-1} X_H^*(l, m)X_H(l, m), \tag{A4}$$

$$\hat{P}_{X_V} = \frac{1}{LM}\sum_{l=0}^{L-1}\sum_{m=0}^{M-1} X_V^*(l, m)X_V(l, m), \tag{A5}$$

$$\hat{R}^{(T)}_{X_H X_V}(0) = \frac{1}{LM}\sum_{l=0}^{L-1}\sum_{m=0}^{M-1} X_V^*(l, m)X_H(l, m), \tag{A6}$$

where L is the oversampling factor, M is the number of pulses, and $X_H(l, m)$ and $X_V(l, m)$ are the whitened signals, as in (3). If the distributions of (A4), (A5), and (A6) are smooth and narrow around their mean values, the variances for $\hat{Z}_{\mathrm{DR}}$, $\hat{\phi}_{\mathrm{DP}}$, and $\hat{\rho}_{\mathrm{HV}}$ can be computed using perturbation analysis (Sachidananda and Zrnić 1985; Ryzhkov and Zrnić 1998; Liu et al. 1994). The results are summarized here:

$$\underset{(\mathrm{WTB})}{\mathrm{Var}\{\hat{Z}_{\mathrm{DR}}\}} = Z^2_{\mathrm{DR}}\left[\mathrm{Var}\left(\frac{\hat{P}_{X_H}}{S_H}\right) + \mathrm{Var}\left(\frac{\hat{P}_{X_V}}{S_V}\right) + 2\,\mathrm{Cov}\left(\frac{\hat{P}_{X_H}}{S_H}, \frac{\hat{P}_{X_V}}{S_V}\right)\right], \tag{A7}$$

$$\underset{(\mathrm{WTB})}{\mathrm{Var}\{\hat{\phi}_{\mathrm{DP}}\}} = \frac{1}{2}\,\mathrm{Re}\left\{E\left[\left|\frac{\hat{R}^{(T)}_{X_H X_V}(0)}{R^{(T)}_{\mathrm{HV}}(0)}\right|^2\right] - E\left[\left(\frac{\hat{R}^{(T)}_{X_H X_V}(0)}{R^{(T)}_{\mathrm{HV}}(0)}\right)^2\right]\right\}, \tag{A8}$$

$$\begin{aligned}\underset{(\mathrm{WTB})}{\mathrm{Var}\{\hat{\rho}_{\mathrm{HV}}\}} = \rho^2_{\mathrm{HV}}\Bigg\langle &\frac{1}{2}\,\mathrm{Re}\left\{E\left[\left(\frac{\hat{R}^{(T)}_{X_H X_V}(0)}{R^{(T)}_{\mathrm{HV}}(0)} - 1\right)^2\right] + E\left[\left|\frac{\hat{R}^{(T)}_{X_H X_V}(0)}{R^{(T)}_{\mathrm{HV}}(0)} - 1\right|^2\right]\right\} \\ &- \mathrm{Re}\left\{\mathrm{Cov}\left[\frac{\hat{P}_{X_H}}{S_H}, \frac{\hat{R}^{(T)}_{X_H X_V}(0)}{R^{(T)}_{\mathrm{HV}}(0)}\right] + \mathrm{Cov}\left[\frac{\hat{P}_{X_V}}{S_V}, \frac{\hat{R}^{(T)}_{X_H X_V}(0)}{R^{(T)}_{\mathrm{HV}}(0)}\right]\right\} + \frac{1}{4}\,\mathrm{Var}\left(\frac{\hat{P}_{X_H}}{S_H}\right) + \frac{1}{4}\,\mathrm{Var}\left(\frac{\hat{P}_{X_V}}{S_V}\right) \\ &+ \frac{1}{2}\,\mathrm{Cov}\left(\frac{\hat{P}_{X_H}}{S_H}, \frac{\hat{P}_{X_V}}{S_V}\right)\Bigg\rangle,\end{aligned} \tag{A9}$$

where S_H and S_V are the true horizontal and vertical channel signal powers, and $R^{(\mathrm{T})}_{\mathrm{HV}}$ is the true cross-correlation function between horizontal and vertical signals. To find these variances the following seven expressions must be evaluated:

$$E\{\hat{P}^2_{X_H}\} = \frac{1}{L^2M^2}\sum_{l=0}^{L-1}\sum_{l'=0}^{L-1}\sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} E[X_H^*(l, m)X_H(l, m)X_H^*(l', m')X_H(l', m')], \tag{A10}$$

$$E\{\hat{P}_{X_V}^2\} = \frac{1}{L^2M^2}\sum_{l=0}^{L-1}\sum_{l'=0}^{L-1}\sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} E[X_V^*(l, m)X_V(l, m)X_V^*(l', m')X_V(l', m')], \quad \text{(A11)}$$

$$E\{\hat{P}_{X_H}\hat{P}_{X_V}\} = \frac{1}{L^2M^2}\sum_{l=0}^{L-1}\sum_{l'=0}^{L-1}\sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} E[X_H^*(l, m)X_H(l, m)X_V^*(l', m')X_V(l', m')], \quad \text{(A12)}$$

$$E\{|\hat{R}_{X_HX_V}^{(T)}(0)|^2\} = \frac{1}{L^2M^2}\sum_{l=0}^{L-1}\sum_{l'=0}^{L-1}\sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} E[X_V^*(l, m)X_H(l, m)X_H^*(l', m')X_V(l', m')], \quad \text{(A13)}$$

$$E\{[\hat{R}_{X_HX_V}^{(T)}(0)]^2\} = \frac{1}{L^2M^2}\sum_{l=0}^{L-1}\sum_{l'=0}^{L-1}\sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} E[X_V^*(l, m)X_H(l, m)X_V^*(l', m')X_H(l', m')], \quad \text{(A14)}$$

$$E\{\hat{P}_{X_H}\hat{R}_{X_HX_V}^{(T)}(0)\} = \frac{1}{L^2M^2}\sum_{l=0}^{L-1}\sum_{l'=0}^{L-1}\sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} E[X_H^*(l, m)X_H(l, m)X_V^*(l', m')X_H(l', m')], \quad \text{(A15)}$$

$$E\{\hat{P}_{X_V}\hat{R}_{X_HX_V}^{(T)}(0)\} = \frac{1}{L^2M^2}\sum_{l=0}^{L-1}\sum_{l'=0}^{L-1}\sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} E[X_V^*(l, m)X_V(l, m)X_V^*(l', m')X_H(l', m')]. \quad \text{(A16)}$$

The first step consist of simplifying the expectation operations inside these expressions using the identity

$$E[X_1^*X_2X_3^*X_4] = E[X_1^*X_2]E[X_3^*X_4] + E[X_1^*X_4]E[X_3^*X_2], \quad \text{(A17)}$$

which is valid for zero-mean, complex, Gaussian random variables (Reed 1962). After applying this identity to Eqs. (A10)–(A16), each expectation can be expressed as the autocorrelation of X at particular lags; for example, $E[X_H^*(l, m)X_H(k, n)] = R_{X_H}(k - l, n - m)$. For example, for (A10)

$$\begin{aligned} E\{\hat{P}_{X_H}^2\} &= \frac{1}{L^2M^2}\sum_{l,l',m,m'} \{E[X_H^*(l, m)X_H(l, m)]E[X_H^*(l', m')X_H(l', m')] \\ &\qquad + E[X_H^*(l, m)X_H(l', m')]E[X_H^*(l', m')X_H(l, m)]\} \\ &= \frac{1}{L^2M^2}\sum_{l,l',m,m'} [R_{X_H}(0, 0)]^2 + R_{X_H}(l' - l, m' - m)R_{X_H}(l - l', m - m'). \end{aligned} \quad \text{(A18)}$$

Then, quadruple summations can be simplified by letting $l'' = l' - l$, $m'' = m' - m$, and collecting terms so that

$$\sum_{l=0}^{L-1}\sum_{l'=0}^{L-1}\sum_{m=0}^{M-1}\sum_{m'=0}^{M-1} f(l' - l, m' - m) = \sum_{l''=-L+1}^{L-1}\sum_{m''=-M+1}^{M-1} (L - |l''|)(M - |m''|)f(l'', m''). \quad \text{(A19)}$$

Following these steps, the following seven expressions arise:

$$E\{\hat{P}_{X_H}^2 - P_H^2\} = \frac{1}{L^2M^2}\sum_{l=-L+1}^{L-1}\sum_{m=-M+1}^{M-1} (L - |l|)(M - |m|)|R_{X_H}(l, m)|^2, \quad \text{(A20)}$$

$$E\{\hat{P}_{X_V}^2 - P_V^2\} = \frac{1}{L^2M^2}\sum_{l=-L+1}^{L-1}\sum_{m=-M+1}^{M-1} (L - |l|)(M - |m|)|R_{X_V}(l, m)|^2, \quad \text{(A21)}$$

$$E\{\hat{P}_{X_H}\hat{P}_{X_V} - P_HP_V\} = \frac{1}{L^2M^2}\sum_{l=-L+1}^{L-1}\sum_{m=-M+1}^{M-1} (L - |l|)(M - |m|)|R_{X_HX_V}(l, m)|^2, \quad \text{(A22)}$$

$$E\{|\hat{R}_{X_HX_V}^{(T)}(0)|^2 - |R_{\mathrm{HV}}^{(T)}(0)|^2\} = \frac{1}{L^2M^2}\sum_{l=-L+1}^{L-1}\sum_{m=-M+1}^{M-1} (L - |l|)(M - |m|)R_{X_H}^*(l, m)R_{X_V}(l, m), \quad \text{(A23)}$$

$$E\{[\hat{R}_{X_HX_V}^{(T)}(0)]^2 - [R_{\mathrm{HV}}^{(T)}(0)]^2\} = \frac{1}{L^2M^2}\sum_{l=-L+1}^{L-1}\sum_{m=-M+1}^{M-1} (L - |l|)(M - |m|)R_{X_HX_V}(l, m)R_{X_HX_V}(-l, -m), \quad \text{(A24)}$$

$$E\{\hat{P}_{X_H}\hat{R}_{X_HX_V}^{(T)}(0) - P_HR_{\mathrm{HV}}^{(T)}(0)\} = \frac{1}{L^2M^2}\sum_{l=-L+1}^{L-1}\sum_{m=-M+1}^{M-1} (L - |l|)(M - |m|)R_{X_HX_V}(-l, -m)R_{X_H}(l, m), \quad \text{(A25)}$$

$$E\{\hat{P}_{X_V}\hat{R}_{X_HX_V}^{(T)}(0) - P_VR_{\mathrm{HV}}^{(T)}(0)\} = \frac{1}{L^2M^2}\sum_{l=-L+1}^{L-1}\sum_{m=-M+1}^{M-1} (L - |l|)(M - |m|)R_{X_HX_V}(l, m)R_{X_V}^*(l, m). \quad \text{(A26)}$$

Because $X_H(l, m)$ and $X_V(l, m)$ have uncorrelated signal and noise components, the autocorrelation function can be decomposed into a sum of the signal (S) and noise (N) autocorrelation functions; for example, $R_{X_H} = R_{X_{H,S}} + R_{X_{H,N}}$. Furthermore, because the dimensions are independent, due to the width of the range-weighting function [Eq. (4.22) of Doviak and Zrnić (1993)] being much smaller than the pulse repetition time, two-dimensional autocorrelation functions can be decomposed in a product of one-dimensional sample-time (T) and range-time (R) autocorrelation functions (see e.g., Dias and Leitão 1993):

$$R_{X_H}(l, m) = R^{(R)}_{X_{H,S}}(l)R^{(T)}_{X_{H,S}}(m) + R^{(R)}_{X_{H,N}}(l)R^{(T)}_{X_{H,N}}(m). \quad \text{(A27)}$$

Double summations can now be decoupled. For instance, (A20) becomes

$$\begin{aligned} E\{\hat{P}^2_{X_H} - P^2_H\} &= \frac{1}{L^2M^2}\sum_{l=-L+1}^{L-1}(L - |l|)|R^{(R)}_{X_{H,S}}(l)|^2 \sum_{m=-M+1}^{M-1}(M - |m|)|R^{(T)}_{X_{H,S}}(m)|^2 \\ &\quad + \frac{2}{L^2M^2}\mathrm{Re}\left\{\sum_{l=-L+1}^{L-1}(L - |l|)R^{(R)}_{X_{H,S}}(l)[R^{(R)}_{X_{H,N}}(l)]^* \sum_{m=-M+1}^{M-1}(M - |m|)R^{(T)}_{X_{H,S}}(m)[R^{(T)}_{X_{H,N}}(m)]^*\right\} \\ &\quad + \frac{1}{L^2M^2}\sum_{l=-L+1}^{L-1}(L - |l|)|R^{(R)}_{X_{H,N}}(l)|^2 \sum_{m=-M+1}^{M-1}(M - |m|)|R^{(T)}_{X_{H,N}}(m)|^2. \end{aligned} \quad \text{(A28)}$$

Assuming that the underlying process V is wide-sense stationary over the oversampled set (L range samples), for a Gaussian sample-time correlation function and white noise $R^{(T)}_{X_S}(m) = S\exp[-2(\pi\sigma_{\upsilon n}m)^2 + j2\pi\upsilon_n m]$ and $R^{(T)}_{X_N}(m) = N\delta(m)$, where $\upsilon_n = \upsilon/2\upsilon_a$ and $\sigma_{\upsilon n} = \sigma_\upsilon/2\upsilon_a$ are the normalized velocity and spectrum width, respectively (Doviak and Zrnić 1993), and υ_a is the maximum unambiguous velocity. If $M\sigma_{\upsilon n} \gg 1$, summations in (A28) involving these functions can be approximated as follows:

$$\sum_{m=-M+1}^{M-1}(M - |m|)|R^{(T)}_{X_{H,S}}(m)|^2 \approx S^2_H\int_{-\infty}^{\infty}(M - |x|)e^{-(2\pi\sigma_{\upsilon n}x)^2}\,dx \approx \frac{MS^2_H}{2\pi^{1/2}\sigma_{\upsilon n}}, \quad \text{(A29)}$$

$$\sum_{m=-M+1}^{M-1}(M - |m|)R^{(T)}_{X_{H,S}}(m)[R^{(T)}_{X_{H,N}}(m)]^* = MS_HN, \quad \text{(A30)}$$

$$\sum_{m=-M+1}^{M-1}(M - |m|)|R^{(T)}_{X_{H,N}}(m)|^2 = MN^2. \quad \text{(A31)}$$

Closed-form solutions for the summations involving range-time correlations can be obtained by working with correlation matrices instead of correlation functions. To make the conversion, the identity $\sum_{l=L+1}^{L-1}(L - |l|)R_1(l)R^*_2(l) = \mathrm{tr}\{\mathbf{C}_1\mathbf{C}_2\}$ can be used,[A1] where $\mathbf{C}_1$ and $\mathbf{C}_2$ are the correlation matrices corresponding to the correlation functions R_1 and R_2, respectively. The relevant terms from (A28) are converted as

$$\sum_{l=-L+1}^{L-1}(L - |l|)|R^{(R)}_{X_{H,S}}(l)|^2 = \mathrm{tr}\{[\mathbf{C}^{(R)}_{X_{H,S}}]^2\}, \quad \text{(A32)}$$

$$\sum_{l=-L+1}^{L-1}(L - |l|)R^{(R)}_{X_{H,S}}(l)[R^{(R}_{X_{H,N}}(l)]^* = \mathrm{tr}\{\mathbf{C}^{(R)}_{X_{H,S}}\mathbf{C}^{(R)}_{X_{H,N}}\}, \quad \text{(A33)}$$

$$\sum_{l=-L+1}^{L-1}(L - |l|)|R^{(R)}_{X_{H,N}}(l)|^2 = \mathrm{tr}\{[\mathbf{C}^{(R)}_{X_{H,N}}]^2\}. \quad \text{(A34)}$$

Correlation matrices for the signal and noise components of the whitened sequence X_H can be obtained from the correlation matrix of the range (correlated) samples V_H by recalling that $\mathbf{X}_H = \mathbf{W}\mathbf{V}_H$ [Eq.(3)], where $\mathbf{W} = \mathbf{H}^{-1}$ and $\mathbf{C}^{(R)}_{V_{H,S}} = \mathbf{H}^*\mathbf{H}^{\mathrm{T}}$. Due to the linear relationship between $\mathbf{V}_H$ and $\mathbf{X}_H$, the correlation matrix of X_H can be written as $\mathbf{C}^{(R)}_{X_H} = \mathbf{W}^*\mathbf{C}^{(R)}_{V_H}\mathbf{W}^{\mathrm{T}}$. Here $\mathbf{C}^{(R)}_{V_H}$ is decomposed into its signal and noise components and distributing the matrix products:

$$\mathbf{C}^{(R)}_{X_H} = \mathbf{W}^*(S\mathbf{C}^{(R)}_{V_{H,S}} + N\mathbf{I})\mathbf{W}^{\mathrm{T}} = S\mathbf{I} + N(\mathbf{W}^*\mathbf{W}^{\mathrm{T}}); \quad \text{(A35)}$$

therefore, $\mathbf{C}^{(R)}_{X_{H,S}} = \mathbf{I}$ and $\mathbf{C}^{(R)}_{X_{H,N}} = \mathbf{W}^*\mathbf{W}^{\mathrm{T}}$. Finally, Eqs. (A32)–(A34) can be written as

$$\mathrm{tr}\{[\mathbf{C}^{(R)}_{X_{H,S}}]^2\} = \mathrm{tr}\{\mathbf{I}^2\} = L, \quad \text{(A36)}$$

$$\begin{aligned} \mathrm{tr}\{\mathbf{C}^{(R)}_{X_{H,S}}\mathbf{C}^{(R)}_{X_{H,N}}\} &= \mathrm{tr}\{\mathbf{W}^*\mathbf{W}^{\mathrm{T}}\} = \mathrm{tr}\{\mathbf{W}^{\mathrm{T}}\mathbf{W}^*\} \\ &= \mathrm{tr}\{[\mathbf{C}^{(R)}_{V_{H,S}}]^{-1}\}, \end{aligned} \quad \text{(A37)}$$

$$\mathrm{tr}\{[\mathbf{C}^{(R)}_{X_{H,N}}]^2\} = \mathrm{tr}\{\mathbf{W}^*\mathbf{W}^{\mathrm{T}}\mathbf{W}^*\mathbf{W}^{\mathrm{T}}\} = \mathrm{tr}\{[\mathbf{C}^{(R)}_{V_{H,S}}]^{-2}\}. \quad \text{(A38)}$$

For the ideal range-time correlation coefficient (corresponding to a rectangular transmitter pulse and an infinite receiver bandwidth) the elements of the received signal correlation matrix are $(\mathbf{C}^{(R)}_{V_{H,S}})_{ij} = 1 - |j - i|L^{-1}$. Due to the structure of this matrix, a closed-form expression for its inverse can be obtained for the general case. It can be verified by matrix multiplication that

[A1] This can be proved by expressing the trace as the sum of diagonal elements of the matrix product, expanding the matrix product using the Hermitian and Toeplitz properties of complex correlation matrices, and finally performing a simple substitution of summation indexes.

$$[\mathbf{C}_{V_{H,S}}^{(R)}]^{-1} = \begin{bmatrix} \frac{L(L+2)}{2L+2} & -\frac{L}{2} & 0 & \cdots & 0 & \frac{L}{2L+2} \\ -\frac{L}{2} & L & -\frac{L}{2} & \ddots & & 0 \\ 0 & -\frac{L}{2} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & L & -\frac{L}{2} \\ \frac{L}{2L+2} & 0 & \cdots & 0 & -\frac{L}{2} & \frac{L(L+2)}{2L+2} \end{bmatrix}. \quad \text{(A39)}$$

Then, $\mathrm{tr}\{[\mathbf{C}_{V_{H,S}}^{(R)}]^{-1}\} = L^3(L+1)^{-1}$. In addition, for symmetric matrices $\mathrm{tr}(A^2) = \Sigma_i \Sigma_f (A)_{ij}^2$ and from (A39) we find that $\mathrm{tr}\{[\mathbf{C}_{V_{H,S}}^{(R)}]^{-2}\} = 1/2L^3(3L^2 + 2L - 3)(L+1)^{-2}$.

The results of (A29)–(A31) and (A36)–(A38) are introduced into (A28):

$$E\{\hat{P}_{X_H}^2 - P_H^2\} = \frac{S_H^2}{2M\sigma_{vn}\sqrt{\pi}}\frac{1}{L} + \frac{2S_H N}{M}\frac{L}{L+1} + \frac{N^2}{M}\frac{L(3L^2+2L-3)}{2(L+1)^2}. \quad \text{(A40)}$$

In a similar way, expressions for Eqs. (A21)–(A26) can be derived to use in Eqs. (A7), (A8), and (A9). Following the steps described above, the simplified expressions for the variances of WTB polarimetric variable estimators are

$$\underset{\text{(WTB)}}{\mathrm{Var}\{\hat{Z}_{\mathrm{DR}}\}} = \frac{Z_{\mathrm{DR}}^2}{M}\left[\left(\frac{1-\rho_{\mathrm{HV}}^2}{\sigma_{vn}\sqrt{\pi}}\right)\frac{1}{L} + 2(1+Z_{\mathrm{DR}})\frac{L}{L+1}\left(\frac{N}{S_H}\right) + (1+Z_{\mathrm{DR}}^2)\frac{L(3L^2+2L-3)}{2(L+1)^2}\left(\frac{N}{S_H}\right)^2\right], \quad \text{(A41)}$$

$$\underset{\text{(WTB)}}{\mathrm{Var}\{\hat{\phi}_{\mathrm{DP}}\}} = \left(\frac{180}{\pi}\right)^2\frac{1}{2M}\left[\left(\frac{\rho_{\mathrm{HV}}^{-2}-11}{2\sigma_{vn}\sqrt{\pi}}\right)\frac{1}{L} + \left(\frac{1+Z_{\mathrm{DR}}}{\rho_{\mathrm{HV}}^2}\right)\frac{L}{L+1}\left(\frac{N}{S_H}\right) + \left(\frac{Z_{\mathrm{DR}}}{\rho_{\mathrm{HV}}^2}\right)\frac{L(3L^2+2L-3)}{2(L+1)^2}\left(\frac{N}{S_H}\right)^2\right], \quad \text{(A42)}$$

$$\underset{\text{(WTB)}}{\mathrm{Var}\{\hat{\rho}_{\mathrm{HV}}\}} = \frac{1}{M}\left\{\left(\frac{1-2\rho_{\mathrm{HV}}^2+\rho_{\mathrm{HV}}^4}{4\sigma_{vn}\sqrt{\pi}}\right)\frac{1}{L} + \left[\frac{(1-\rho_{\mathrm{HV}}^2)(1+Z_{\mathrm{DR}})}{2}\right]\frac{L}{L+1}\left(\frac{N}{S_H}\right) + \left(\frac{\rho_{\mathrm{HV}}^2+2Z_{\mathrm{DR}}+\rho_{\mathrm{HV}}^2Z_{\mathrm{DR}}^2}{4}\right)\frac{L(3L^2+2L-3)}{2(L+1)^2}\left(\frac{N}{S_H}\right)^2\right\}. \quad \text{(A43)}$$

The same procedure can be repeated for the case of a digital matched filter where the only difference with the previous case is in the way that the total powers for the horizontal and vertical channels and lag-zero cross-correlation function are estimated. For the matched-filter case

$$\hat{P}_{Y_H} = \frac{1}{M}\sum_{m=0}^{M-1}|Y_H(m)|^2, \quad \text{(A44)}$$

$$\hat{P}_{Y_V} = \frac{1}{M}\sum_{m=0}^{M-1}|Y_V(m)|^2, \quad \text{(A45)}$$

$$\hat{R}_{Y_H Y_V}^{(T)}(0) = \frac{1}{M}\sum_{m=0}^{M-1}Y_V^*(m)Y_H(m), \quad \text{(A46)}$$

where $Y(m) = \kappa\,\Sigma_{l=0}^{L-1}\,V(l, m)$ is the output of the digital matched filter, and $\kappa = \sqrt{3/(2L^2+1)}$ is a normalization factor that preserves the power of the input signal under ideal conditions. Following the same steps as outlined for the previous case, the variances of the matched-filter-based estimators are obtained as

$$\underset{\text{(MFB)}}{\mathrm{Var}\{\hat{Z}_{\mathrm{DR}}\}} = \frac{Z_{\mathrm{DR}}^2}{M}\left[\left(\frac{1-\rho_{\mathrm{HV}}^2}{\sigma_{vn}\sqrt{\pi}}\right) + 2(1+Z_{\mathrm{DR}})\frac{3L}{2L^2+1}\left(\frac{N}{S_H}\right) + (1+Z_{\mathrm{DR}}^2)\left(\frac{3L}{2L^2+1}\right)^2\left(\frac{N}{S_H}\right)^2\right], \quad \text{(A47)}$$

$$\underset{\text{(MFB)}}{\mathrm{Var}\{\hat{\phi}_{\mathrm{DP}}\}} = \left(\frac{180}{\pi}\right)^2\frac{1}{2M}\left[\left(\frac{\rho_{\mathrm{HV}}^{-2}-1}{2\sigma_{vn}\sqrt{\pi}}\right) + \left(\frac{1+Z_{\mathrm{DR}}}{\rho_{\mathrm{HV}}^2}\right)\frac{3L}{2L^2+1}\left(\frac{N}{S_H}\right) + \left(\frac{Z_{\mathrm{DR}}}{\rho_{\mathrm{HV}}^2}\right)\left(\frac{3L}{2L^2+1}\right)^2\left(\frac{N}{S_H}\right)^2\right], \quad \text{(A48)}$$

$$\underset{(\mathrm{MFB})}{\mathrm{Var}\{\hat{\rho}_{\mathrm{HV}}\}} = \frac{1}{M}\left\{\left(\frac{1 - 2\rho_{\mathrm{HV}}^2 + \rho_{\mathrm{HV}}^4}{4\sigma_{vn}\sqrt{\pi}}\right) + \left[\frac{(1 - \rho_{\mathrm{HV}}^2)(1 + Z_{\mathrm{DR}})}{2}\right]\frac{3L}{2L^2 + 1}\left(\frac{N}{S_H}\right) + \left(\frac{\rho_{\mathrm{HV}}^2 + 2Z_{\mathrm{DR}} + \rho_{\mathrm{HV}}^2 Z_{\mathrm{DR}}^2}{4}\right)\left(\frac{3L}{2L^2 + 1}\right)^2\left(\frac{N}{S_H}\right)^2\right\}. \quad \text{(A49)}$$

APPENDIX B

Simulation of Oversampled Polarimetric Weather Echoes

This appendix describes a simulation method that produces a dual-polarization pair of oversampled time-series data V_H and V_V. The two time series exhibit the required marginal structure, that is, correlation in both range and sample time. Simultaneously, it is required to specify the properties of the joint density through the cross-correlation function $R^{(T)}_{V_H V_V}(m)$. First, correlation along range time is imposed by dividing the resolution volume into ''slabs'' and by weighting the contributions from each slab using the proper range-weighting function. Second, the well-known procedure for generating single-polarization time series by Zrnić (1975) is used to shape the autocorrelation along sample time. Finally, the procedure by Galati and Pavan (1995) for generating a pair of correlated time series is used to impose the cross-correlation between horizontal and vertical channel time series. Note that for bivariate Gaussian processes, the auto- and cross-correlation functions (or their equivalent power spectral densities) provide a complete description of the underlying processes. The first two steps of the simulation are described in detail in Torres and Zrnić (2003) for single-polarized weather echoes. The third step, which is specific to the generation of dual-polarized time-series data, is described next.

Let X and Y be unit-power, independent time series with the required marginal range- and sample-time correlations [$\rho^{(R)}$ and $\rho^{(T)}$]. The signals from the horizontal and vertical channels can be constructed by means of the following transformation:

$$V_H(l, n) = \gamma X(l, n)$$
$$V_V(l, n) = \alpha X(l, n) + \beta Y(l, n), \quad \text{(B1)}$$

where γ, α, and β are complex constants to be determined. It is not difficult to verify that this transformation produces the desired result. In the case of V_H, it is obvious that the range- and sample-time correlations are the prescribed ones:

$$R^{(R)}_{V_H}(m) = E[V^*_H(l, n)V_H(l + m, n)]/R^{(T)}_{V_H}(0) = |\gamma|^2\rho^{(R)}(m)/S_H, \quad \text{(B2)}$$

$$R^{(T)}_{V_H}(m) = E[V^*_H(l, n)V_H(l, n + m)] = |\gamma|^2\rho^{(T)}(m). \quad \text{(B3)}$$

The required marginal correlations of X are clearly transferred into V_H. Further, since X has unit power, (B2) and (B3) give

$$|\gamma|^2 = S_H. \quad \text{(B4)}$$

For V_V, it follows that

$$R^{(R)}_{V_V}(m) = E[V^*_V(l, n)V_V(l + m, n)]/R^{(T)}_{V_V}(0) = [|\alpha|^2 R^{(R)}_X(m) + |\beta|^2 R^{(R)}_Y(m)]/S_V = (|\alpha|^2 + |\beta|^2)\rho^{(R)}(m)/S_V, \quad \text{(B5)}$$

$$R^{(T)}_{V_V}(m) = E[V^*_V(l, n)V_V(l, n + m)] = |\alpha|^2 R^{(T)}_X(m) + |\beta|^2 R^{(T)}_Y(m) = (|\alpha|^2 + |\beta|^2)\rho^{(T)}(m). \quad \text{(B6)}$$

Again, the required marginal correlation is transferred to V_V, and to obtain the required correlation in range

$$|\alpha|^2 + |\beta|^2 = S_V. \quad \text{(B7)}$$

The cross-correlation between V_H and V_V is

$$R^{(T)}_{V_H V_V}(m) = E[V^*_V(l, n)V_H(l, n + m)] = \gamma\alpha^*\rho^{(T)}(m), \quad \text{(B8)}$$

and since $R^{(T)}_{V_H V_V}(m) = R^{(T)}_{V_H V_V}(0)\rho^{(T)}(m)$ (Sachidananda and Zrnić 1986) it follows that

$$\gamma\alpha^* = R^{(T)}_{V_H V_V}(0). \quad \text{(B9)}$$

Summarizing, constants γ, α, and β must be determined from Eqs. (B4), (B7), and (B9). This is an underdetermined set of equations, so γ can be chosen to be real to get $\gamma = \sqrt{S_H}$ and $\alpha = [R^{(T)}_{V_H V_V}(0)]^*/\sqrt{S_H} = \sqrt{S_V}\rho_{\mathrm{HV}}e^{-j\phi_{\mathrm{DP}}}$. Finally, one can solve for $|\beta|$ and choose its argument to be the same as α's to get $\beta = \sqrt{S_V(1 - \rho^2_{\mathrm{HV}})}e^{-j\phi_{\mathrm{DP}}}$. Then, the transformation becomes

$$V_H(l, n) = \sqrt{S_H}X(l, n)$$
$$V_V(l, n) = \sqrt{S_V}\left[\rho_{\mathrm{HV}}X(l, n) + \sqrt{1 - \rho^2_{\mathrm{HV}}}\,Y(l, n)\right]e^{-j\phi_{\mathrm{DP}}}. \quad \text{(B10)}$$

A different technique for generating dual-polarized echoes was introduced by Chandrasekar and Bringi (1986). This approach makes use of the structure of the covariance matrix of the spectral components of a bivariate Gaussian time series and is intrinsically a frequency-domain method. Consequently, it is computationally more intensive than the procedure described by

Galati and Pavan (1995). Although both methods yield equivalent results, it turns out that Chandrasekar's approach is more general because it can simulate bivariate Gaussian random processes with arbitrary auto- and cross-correlation functions. On the other hand, the method described by (B10) requires that the two random processes exhibit the same sample-time autocorrelation and that the cross-correlation be proportional to the marginal autocorrelation. In other words, the method works well only if $R^{(T)}_{V_V} = R^{(T)}_{V_H} = \kappa R^{(T)}_{V_H V_V}$, which holds true for the case of horizontally and vertically polarized weather echoes (Sachidananda et al. 1986).

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