Processing of Oversampled Signals in Range on Polarimetric Weather Radars with Mismatched Channels

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ABSTRACT

Processing oversampled signals in range with a whitening transformation has been proposed as a means to reduce the variance of meteorological variable estimates on polarimetric Doppler weather radars. However, the original formulation to construct decorrelation transformations does not account for mismatches in the polarimetric channels, which results in abnormally biased polarimetric variable estimates if the two channels are not perfectly matched. This paper extends the initial formulation and demonstrates that, by properly accounting for the differences in the polarimetric channels, it is always possible to produce optimum estimates of all meteorological variables. Simulation analyses based on the reported characteristics of existing polarimetric radars are included to illustrate the performance of the proposed transformations.

1. Introduction

Range oversampling followed by a decorrelation transformation is a recently suggested method for increasing the number of independent samples from which to estimate the Doppler spectrum and its moments, as well as several polarimetric variables on pulsed weather radars (Torres and Zrnić 2003a,b). Range-oversampling techniques rely on the precise knowledge of the range correlation of oversampled signals, which is a function of the transmitter pulse envelope, the receiver filter impulse response, and the distribution of scatterers illuminated by the radar. Theoretical and simulation studies demonstrating the advantages of these techniques have been successfully verified on weather data collected with a single-transmitter dual-polarization radar (Ivić et al. 2002; Torres and Ivić 2005). In contrast, recent experimental results on a dual-transmitter system have revealed some difficulties: if the amplitude and/or phase mismatch between transmission pulses is disregarded in the formulation of the decorrelation transformation, processing of range-oversampled dual-polarization signals with the standard whitening transformation can produce abnormally biased[1] polarimetric variable estimates (Choudhury and Chandrasekar 2007). These authors concluded that matching the correlation of samples in range for the horizontal and vertical channels is critical to effectively use the whitening transformation introduced by Torres and Zrnić (2003b). Further, they recognized "the need to develop a variant of the whitening transformation algorithm" for dual-polarization systems with mismatched channels. Recently, Hefner and Chandrasekar (2008) proposed a Hermitian symmetric whitening transformation as a means to mitigate the biases observed when applying the originally proposed whitening transformation. In addition, a Hermitian transformation was shown to fix numerical inconsistencies that could arise in the construction of whitening transformations from mismatched polarimetric channels. Although these results are very encouraging, the proposed whitening transformation does not remove the biases completely and the authors conclude that more work is needed to determine unbiased whitening (UWTB) methods. This is the purpose of this work.

[1] Throughout this work, the term abnormally biased is used to denote range-oversampling polarimetric variable estimators having biases larger than their standard (no oversampling) counterparts. Range-oversampling estimators with the same or smaller biases than their standard counterparts are termed normally biased.

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Although having a dual-polarization radar system with matched channels is ideal for proper measurement of the polarimetric variables, there are situations in which, despite the best design efforts, one must deal with a system with mismatched polarization channels. Systems with one transmitter and a power splitter, such as the National Severe Storms Laboratory KOUN radar (Zahrai and Zrnić 1993), are relatively immune to mismatches in the horizontal and vertical channels, as demonstrated by Torres and Ivić (2005). Still, differences may arise because of variations in the hardware paths specific to each channel. In contrast, systems with dual transmitters, such as the Colorado State University–University of Chicago–Illinois State Water Survey (CSU–CHILL) radar (Brunkow et al. 2000), are more susceptible to waveform mismatches, as reported by Choudhury and Chandrasekar (2007). This problem is aggravated with magnetron-based radars, because precise control of transmitted waveform phases and frequencies is not possible. Despite this limitation, dual-transmitter systems may be preferred as a way to increase the sensitivity of the radar or to exploit the ability to control each transmitted waveform individually, such as required by the method suggested by Chandrasekar et al. (2007) to simultaneously perform co- and cross-polarization measurements.

This work demonstrates that, by properly accounting for the amplitude and/or phase differences in the two polarization channels, it is always possible to obtain acceptable polarimetric variable estimates from transformed range-oversampled data. Nonetheless, the variance of these estimators increases as the degree of mismatch between the horizontally and the vertically polarized transmitted pulses increases. In such cases, estimators that achieve maximum variance reduction can be obtained by solving constrained minimization problems.

The paper is organized as follows: section 2 reviews the theory behind range-oversampling techniques on dual-polarimetric radars. Section 3 examines the bias in auto- and cross-correlation estimates for systems with matched channels. This is followed by a similar analysis for systems with mismatched channels in section 4, in which a formulation that leads to normally biased estimates of the polarimetric variables is presented. Section 5 discusses the construction of optimum transformations for each of the polarimetric variables. The final section demonstrates the performance of the different transformations using simulated data.

2. Range oversampling in dual-polarimetric radars

Traditional sampling of weather radar signals V occurs at a rate of τ^{-1}, where τ is the duration of the transmitted pulse. Oversampling in range entails acquiring polarimetric time series data at increased rates so that L complex samples are collected during the time τ. This is termed as oversampling by a factor of L and has become feasible with the advent of commercial single-board digital receivers (Ivić et al. 2003a) and digital signal processors (Zahrai et al. 2002).

a. Characterization of range-oversampled dual-polarimetric signals

Let $\mathbf{v}_H$ and $\mathbf{v}_V$ be the sets of L oversampled signals in range for the horizontal (H) and vertical (V) polarization channels for a given sample time m. In vector notation,

$$\mathbf{v}_{H,V} = [V_{H,V}(0,m)\ V_{H,V}(1,m)\ \ldots\ V_{H,V}(L-1,m)]^{\mathrm{T}}, \tag{1}$$

where the superscript T denotes matrix transposition and the subscript H, V (read as H or V) denote signals corresponding to either the horizontal or vertical channels. The first index in the time series corresponds to range time; the second corresponds to sample time. The two-dimensional correlation of range-oversampled signals considering both range- and sample-time lags is defined as a separable function:

$$\begin{aligned} R_{V_YV_Z}(k,n) &= E[V_Y^*(l,m)V_Z(l+k,m+n)] \\ &= \rho^{(R)}_{V_YV_Z}(k)R^{(T)}_{V_YV_Z}(n), \end{aligned} \tag{2}$$

where k and n are range- and sample-time lags, $E[.]$ is the expected value operation, the superscript * denotes complex conjugation, superscripts (R) and (T) designate range- and sample-time correlations, and subscripts Y and Z can be either H or V denoting signals from the horizontal or vertical channels (e.g., $\rho^{(R)}_{V_HV_H}$ is the range-time autocorrelation for the horizontal channel and $\rho^{(R)}_{V_HV_V}$ is the range-time cross correlation between the horizontal and vertical channels). If the resolution volume is uniformly filled with scatterers and the effects of receiver noise are ignored, the correlation coefficient of the oversampled range samples is solely determined by the transmitted pulse shape and the receiver filter impulse response. Let p_H and p_V be the normalized "modified" pulse envelopes for the horizontal and vertical channels (i.e., the transmitted pulses after each channel's receiver filter). For a calibrated system, the amplitudes of the pulses are such that they do not bias the power in each channel; that is, $\Sigma_{l=0}^{L-1}|p_{H,V}(l)|^2 = 1$. Then, the correlation coefficient for range samples can be obtained as (Torres and Zrnić 2003a)

$$\rho^{(R)}_{V_YV_Z}(l) = p_Y^*(l) * p_Z(-l), \tag{3}$$

where * is the convolution operation. From this, normalized range correlation matrices can be constructed as

$$\{\mathbf{C}_{V_Y V_Z}\}_{i,j} = \rho^{(R)}_{V_Y V_Z}(j-i), \tag{4}$$

where $\{\mathbf{C}\}_{i,j}$ denotes the element in the ith row and jth column of the Hermitian matrix $\mathbf{C}$.

b. Estimation of auto- and cross correlations

Oversampled signals in range can be used to improve the quality of meteorological variable estimates without increasing volume acquisition times. Because the goal is to produce better-quality estimates for the traditional (nonoversampled) range gate spacing, a set of signals at L oversampled range gates are suitably combined. With this technique, auto- and cross correlations are estimated at each of the L oversampled range gates. These L correlation estimates are averaged to produce one correlation estimate with reduced variance. As with traditional sampling, averaged auto- and cross correlations for the first few lags are used to compute the spectral moments and the polarimetric variables. The focus of this paper is on the estimation of the polarimetric variables: differential reflectivity Z_{DR}, differential phase Φ_{DP}, and magnitude of the cross-correlation coefficient ρ_{HV}. Classical estimators are given by

$$\hat{Z}_{\mathrm{DR}} = \frac{\hat{R}_{V_H V_H}(0)}{\hat{R}_{V_V V_V}(0)}, \tag{5}$$

$$\hat{\Phi}_{\mathrm{DP}} = \arg[\hat{R}_{V_H V_V}(0)], \quad \text{and} \tag{6}$$

$$\hat{\rho}_{HV} = \frac{\hat{R}_{V_H V_V}(0)}{\sqrt{\hat{R}_{V_H V_H}(0)\hat{R}_{V_V V_V}(0)}}, \tag{7}$$

where the "hat" is used to denote an estimate. Hence, only zero-lag auto- and cross correlations need to be examined, and the lag indexing will be dropped from the notation for simplicity. Sample-time zero-lag auto- and cross-correlation estimates from oversampled signals are given by

$$\hat{R}^{(T)}_{V_Y V_Z} = \frac{1}{ML}\sum_{l=0}^{L-1}\sum_{m=0}^{M-1} V_Y^*(l,m)V_Z(l,m), \tag{8}$$

where M is the number of samples in the dwell time and Y and Z can again be either H or V. Equation (8) can be rewritten as

$$\hat{R}^{(T)}_{V_Y V_Z} = \frac{1}{M}\sum_{m=0}^{M-1}\left[\frac{1}{L}\sum_{l=0}^{L-1} V_Y^*(l,m)V_Z(l,m)\right], \tag{9}$$

where it is more evident that it is possible to produce correlation estimates with lower variance by reducing the variance of range-averaged correlations. In other words, we would like to transform range-oversampled signals to produce uncorrelated data that can be exploited to maximize the variance reduction through averaging (Torres and Zrnić 2003a). In addition, this transformation must result in unbiased correlation estimates to preserve (in average) the integrity of the polarimetric variables (see the appendix).

The expected value of (9) is

$$E[\hat{R}^{(T)}_{V_Y V_Z}] = \frac{1}{ML}\sum_{l=0}^{L-1}\sum_{m=0}^{M-1} E[V_Y^*(l,m)V_Z(l,m)] = R_{V_Y V_Z}. \tag{10}$$

Using (2) and (4),

$$E[\hat{R}^{(T)}_{V_Y V_Z}] = \rho^{(R)}_{V_Y V_Z} R^{(T)}_{V_Y V_Z} = \frac{\mathrm{tr}(\mathbf{C}_{V_Y V_Z})}{L} R^{(T)}_{V_Y V_Z}, \tag{11}$$

where tr(.) is the trace of a matrix. The normalized bias of the correlation estimator is

$$\frac{\mathrm{Bias}[\hat{R}^{(T)}_{V_Y V_Z}]}{R^{(T)}_{V_Y V_Z}} = \frac{E[\hat{R}^{(T)}_{V_Y V_Z}] - R^{(T)}_{V_Y V_Z}}{R^{(T)}_{V_Y V_Z}} = \frac{\mathrm{tr}(\mathbf{C}_{V_Y V_Z})}{L} - 1, \tag{12}$$

where it is evident that unbiased correlation estimates require $\mathrm{tr}(\mathbf{C}_{V_Y V_Z}) = L$.

c. Transformation of range-oversampled signals

A whitening transformation on range-oversampled time series data can be used to decorrelate these signals before averaging; that is, through a linear transformation, a set of L correlated complex samples is transformed into a set of L decorrelated (or whitened) complex samples. Because data are uncorrelated, averaging covariances after whitening oversampled signals reduces the variance of estimates by a factor of L (Torres and Zrnić 2003a).

The whitening transformation $\mathbf{W}$ can be constructed as

$$\mathbf{W} = \mathbf{H}^{-1}, \tag{13}$$

where $\mathbf{H}$ comes from the square-root decomposition of the normalized autocorrelation matrix; that is, $\mathbf{C} = \mathbf{H}^*\mathbf{H}^{\mathrm{T}}$. It is important to note that this decomposition is not unique. As argued by Torres et al. (2004), a family of whitening transformations can be obtained by premultiplying the inverse of a given matrix square root of $\mathbf{C}$ with any unitary matrix. This fact was exploited by

Hefner and Chandrasekar (2008) to construct Hermitian symmetric whitening transformations based on the eigenvalue decomposition of $\mathbf{C}$.

Regardless of the $\mathbf{H}$ matrix used in (13), a vector $\mathbf{x}$ of L transformed oversampled data at a given sample time is obtained as

$$\mathbf{x}_{H,V} = \mathbf{W}\mathbf{v}_{H,V}, \tag{14}$$

and range correlation matrices for the transformed data are

$$\begin{aligned}\mathbf{C}_{X_Y X_Z} &= E[\mathbf{x}_Y^* \mathbf{x}_Z^{\mathrm{T}}] = \mathbf{W}^* E[\mathbf{v}_Y^* \mathbf{v}_Z^{\mathrm{T}}]\mathbf{W}^{\mathrm{T}} \\ &= \mathbf{W}^* \mathbf{C}_{V_Y V_Z} \mathbf{W}^{\mathrm{T}},\end{aligned} \tag{15}$$

where Y and Z can again be either H or V.

3. Range oversampling on systems with matched polarimetric channels

For a radar system with perfectly matched channels, the normalized modified pulses for the H and V channels are the same; that is, $p_H = p_V$. This is not an unrealistic assumption for dual-polarization radars with one transmitter. In this situation, the normalized auto- and cross-correlation matrices are the same [cf. (3)]; that is, $\mathbf{C}_{V_H V_H} = \mathbf{C}_{V_V V_V} = \mathbf{C}_{V_H V_V}$.

a. Autocorrelation estimation

Using (12), the normalized bias of the autocorrelation estimator on transformed data is

$$\frac{\mathrm{Bias}[\hat{R}^{(T)}_{X_H X_H}]}{R^{(T)}_{X_H X_H}} = \frac{\mathrm{tr}(\mathbf{C}_{X_H X_H})}{L} - 1 = \frac{\mathrm{tr}(\mathbf{W}^* \mathbf{C}_{V_H V_H} \mathbf{W}^{\mathrm{T}})}{L} - 1. \tag{16}$$

The matrix product inside the trace can be simplified using (13) and a square-root decomposition of $\mathbf{C}_{V_H V_H}$ as

$$\begin{aligned}\mathbf{W}^* \mathbf{C}_{V_H V_H} \mathbf{W}^{\mathrm{T}} &= (\mathbf{H}^{-1})^* \mathbf{H}^* \mathbf{H}^{\mathrm{T}} (\mathbf{H}^{-1})^{\mathrm{T}} \\ &= (\mathbf{H}^{-1}\mathbf{H})^* (\mathbf{H}^{-1}\mathbf{H})^{\mathrm{T}} = \mathbf{I},\end{aligned} \tag{17}$$

where $\mathbf{I}$ is the identity matrix (i.e., transformed data are uncorrelated). Hence, (16) becomes

$$\frac{\mathrm{Bias}[\hat{R}^{(T)}_{X_H X_H}]}{R^{(T)}_{X_H X_H}} = \frac{\mathrm{tr}(\mathbf{I})}{L} - 1 = 0, \tag{18}$$

and the autocorrelation estimator on transformed data is unbiased with the transformation defined by (13). The same is true for the V channel autocorrelation estimator on transformed data, because $\mathbf{C}_{V_H V_H} = \mathbf{C}_{V_V V_V}$ and $\mathbf{W}$ also whitens the V channel data.

b. Cross-correlation estimation

Similarly,

$$\begin{aligned}\frac{\mathrm{Bias}[\hat{R}^{(T)}_{X_H X_V}]}{R^{(T)}_{X_H X_V}} &= \frac{\mathrm{tr}(\mathbf{C}_{X_H X_V})}{L} - 1 = \frac{\mathrm{tr}(\mathbf{W}^* \mathbf{C}_{V_H V_V} \mathbf{W}^{\mathrm{T}})}{L} - 1 \\ &= \frac{\mathrm{tr}(\mathbf{I})}{L} - 1 = 0.\end{aligned} \tag{19}$$

Thus, the cross-correlation estimator on transformed data with the transformation defined by (13) is also unbiased because $\mathbf{C}_{V_H V_H} = \mathbf{C}_{V_H V_V}$.

4. Range oversampling on systems with mismatched polarimetric channels

For a radar system with mismatched channels, the normalized modified pulses for the H and V channels are different; that is, $p_H \neq p_V$. This is more likely to occur in dual-polarization radars with dual transmitters. In this case, auto- and cross-correlation matrices are generally different. Therefore, a whitening matrix that works for the H channel may not work for the V channel and vice versa. Then, it makes sense to consider two independent whitening transformations, $\mathbf{W}_H$ and $\mathbf{W}_V$—one for each channel. Transformed data $\mathbf{x}$ are obtained as [cf. (14)]

$$\mathbf{x}_{H,V} = \mathbf{W}_{H,V}\mathbf{v}_{H,V}, \tag{20}$$

and normalized range correlation matrices are [cf. (15)]

$$\mathbf{C}_{X_Y X_Z} = \mathbf{W}_Y^* \mathbf{C}_{V_Y V_Z} \mathbf{W}_Z^{\mathrm{T}}. \tag{21}$$

a. Autocorrelation estimation

Repeating the process in the previous section, the normalized bias of the autocorrelation estimator on transformed data is

$$\begin{aligned}\frac{\mathrm{Bias}[\hat{R}^{(T)}_{X_H X_H}]}{R^{(T)}_{X_H X_H}} &= \frac{\mathrm{tr}(\mathbf{C}_{X_H X_H})}{L} - 1 = \frac{\mathrm{tr}(\mathbf{W}_H^* \mathbf{C}_{V_H V_H} \mathbf{W}_H^{\mathrm{T}})}{L} - 1 \\ &= \frac{\mathrm{tr}(\mathbf{I})}{L} - 1 = 0,\end{aligned} \tag{22}$$

hence the estimator is unbiased with $\mathbf{W}_H$ defined by (13). The same is true for the V channel autocorrelation estimator, with $\mathbf{W}_V$ derived from an analogous decomposition of $\mathbf{C}_{V_V V_V}$.

b. Biased cross-correlation estimation

A similar analysis as above reveals that

$$\frac{\mathrm{Bias}[\hat{R}^{(T)}_{X_H X_V}]}{R^{(T)}_{X_H X_V}} = \frac{\mathrm{tr}(\mathbf{C}_{X_H X_V})}{L} - 1 = \frac{\mathrm{tr}(\mathbf{W}^*_H \mathbf{C}_{V_H V_V} \mathbf{W}^{\mathrm{T}}_V)}{L} - 1. \tag{23}$$

Hence, the cross-correlation estimator is biased because, in general, $\mathrm{tr}(\mathbf{W}^*_H \mathbf{C}_{V_H V_V} \mathbf{W}^T_V) \neq L$. A simple example suffices to demonstrate this problem. Consider the case of mismatched H and V channels such that $p_V = e^{j\alpha} p_H$, where α is a real number. Then, according to (3), $\mathbf{C}_{V_V V_V} = \mathbf{C}_{V_H V_H} = \mathbf{H}^* \mathbf{H}^{\mathrm{T}}$ and $\mathbf{C}_{V_H V_V} = e^{j\alpha} \mathbf{C}_{V_H V_H}$. If $\mathbf{W}_H = \mathbf{W}_V = \mathbf{H}^{-1}$ are the whitening transformations for each channel, it is easy to see that

$$\begin{aligned} \mathrm{tr}(\mathbf{W}^*_H \mathbf{C}_{V_H V_V} \mathbf{W}^{\mathrm{T}}_V) &= \mathrm{tr}[(\mathbf{H}^{-1})^* (e^{j\alpha} \mathbf{H}^* \mathbf{H}^{\mathrm{T}})(\mathbf{H}^{-1})^{\mathrm{T}}] \\ &= \mathrm{tr}(e^{j\alpha} \mathbf{I}) = L e^{j\alpha}, \end{aligned} \tag{24}$$

and these whitening transformations would result in unbiased cross-correlation estimates only if α is an integer multiple of 2π.

c. Unbiased cross-correlation estimation

The result in (23) is useful for constructing transformations that lead to unbiased cross-correlation estimates. As shown before, the condition for unbiased estimates is given by

$$\mathrm{tr}(\mathbf{W}^*_H \mathbf{C}_{V_H V_V} \mathbf{W}^{\mathrm{T}}_V) = L; \tag{25}$$

this is easily achieved by properly scaling the H and V transformation matrices; that is, let a new set of scaled transformation matrices for the cross-correlation estimator be $\tilde{\mathbf{W}}_{H,V} = \tilde{\gamma}_{H,V} \mathbf{W}_{H,V}$, where $\tilde{\gamma}_{H,V}$ are complex constants (the tilde is used throughout to discriminate the cross-correlation estimator transformations from the ones corresponding to the autocorrelation estimator). With these new transformations, (25) becomes

$$\tilde{\gamma}^*_H \tilde{\gamma}_V \mathrm{tr}(\mathbf{W}^*_H \mathbf{C}_{V_H V_V} \mathbf{W}^{\mathrm{T}}_V) = L \tag{26}$$

and the scaling factors must be chosen such that

$$\tilde{\gamma}^*_H \tilde{\gamma}_V = \frac{L}{\mathrm{tr}(\mathbf{W}^*_H \mathbf{C}_{V_H V_V} \mathbf{W}^{\mathrm{T}}_V)}. \tag{27}$$

A solution to this equation is

$$\tilde{\gamma}^*_H = \tilde{\gamma}_V = \sqrt{\frac{L}{\mathrm{tr}(\mathbf{W}^*_H \mathbf{C}_{V_H V_V} \mathbf{W}^{\mathrm{T}}_V)}}. \tag{28}$$

Note that with this formulation, the transformation matrices used in the autocorrelation estimators are scaled differently from the ones used in the cross-correlation estimator, but their basic structure is the same (i.e., both sets are based on whitening transformations for each channel).

d. General unbiased correlation estimation

As will be shown in section 6, scaled transformations based on whitening transformations for each channel may not result in polarimetric variable estimators with the lowest possible variance. Thus, it may be advantageous to explore other transformation structures. Other transformations such as pseudowhitening have been proposed as a way to increase the effective number of independent samples while minimizing the noise enhancement effects inherent in the whitening transformation (Torres et al. 2004). In general, we need a procedure to determine the best transformation matrices that result in unbiased correlation estimates for any given situation without being constrained to choosing whitening transformations. The general formulation below produces unbiased auto- and cross-correlation estimates for any transformation matrix structure.

Let, $\mathbf{A}_{H,V}$ be the set of transformations for the autocorrelation estimator, and $\tilde{\mathbf{A}}_{H,V}$ be the set for the cross-correlation estimator, where the basic structure of each matrix can be determined using different criteria (e.g., $\mathbf{A}_{H,V}$ can be chosen as the whitening matrices for the H and V channel data, whereas $\tilde{\mathbf{A}}_{H,V}$ can be chosen as the matrices that diagonalize the range cross-correlation matrix $\mathbf{C}_{V_V V_H}$). To produce unbiased auto- and cross-correlation estimates, these four matrices require scaling given by respective factors, which could be determined as

$$\gamma_{H,V} = \sqrt{\frac{L}{\mathrm{tr}(\mathbf{A}^*_{H,V} \mathbf{C}_{V_{H,V} V_{H,V}} \mathbf{A}^T_{H,V})}} \quad \text{and} \tag{29}$$

$$\tilde{\gamma}^*_H = \tilde{\gamma}_V = \sqrt{\frac{L}{\mathrm{tr}(\tilde{\mathbf{A}}^*_H \mathbf{C}_{V_H V_V} \tilde{\mathbf{A}}^T_V)}}. \tag{30}$$

Hence, scaled transformations are $\mathbf{W}_{H,V} = \gamma_{H,V} \mathbf{A}_{H,V}$ and $\tilde{\mathbf{W}}_{H,V} = \tilde{\gamma}_{H,V} \tilde{\mathbf{A}}_{H,V}$. Note that with this scaling, auto- and cross-correlation estimates are always unbiased. However, the variance of these estimators depends on the basic structure chosen for each transformation matrix.

5. Optimum unbiased correlation estimation

A general way to construct transformations that lead to unbiased correlation estimators was presented in the previous section. The next logical step is to find the optimum set of transformations that produces unbiased correlation estimates and leads to polarimetric variables with the lowest variance. In general, for a polarimetric variable estimator $\hat{\theta}$, where θ is a function of one or more correlation estimates, we must find the solution to the following constrained minimization problem:

$$\min_{\mathbf{W}_H,\mathbf{W}_V,\tilde{\mathbf{W}}_H,\tilde{\mathbf{W}}_V} \mathrm{Var}[\hat{\theta}]$$

subject to

$$\mathrm{tr}(\mathbf{W}_H^*\mathbf{C}_{V_HV_H}\mathbf{W}_H^{\mathrm{T}}) = L, \quad \mathrm{tr}(\mathbf{W}_V^*\mathbf{C}_{V_VV_V}\mathbf{W}_V^{\mathrm{T}}) = L, \quad \text{and}$$
$$\mathrm{tr}(\tilde{\mathbf{W}}_H^*\mathbf{C}_{V_HV_V}\tilde{\mathbf{W}}_V^{\mathrm{T}}) = L, \tag{31}$$

where $\mathrm{Var}[\hat{\theta}]$ is a function of one or more transformation matrices $\mathbf{W}_H, \mathbf{W}_V, \tilde{\mathbf{W}}_H,$ and $\tilde{\mathbf{W}}_V$. Note that the constraint in the minimization problem guarantees polarimetric variable estimates with acceptable biases (see the appendix). As presented in the previous section, this constraint is easily satisfied by the scaling in Eqs. (29) and (30) for generic (not necessarily whitening) transformation matrices $\mathbf{W}_{H,V}$ and $\tilde{\mathbf{W}}_{H,V}$ corresponding to the auto- and cross-correlation estimators, respectively.

The solution to the problem in (31) depends on the theoretical form that the variance of polarimetric variable estimates takes as a function of the transformation matrices. Hence, the first step to finding the solution for a particular estimator is to express its variance as a function of the normalized auto- and cross-correlation matrices given in (21), which will explicitly show the dependency on the oversampled data transformations. In general, variances of the polarimetric variables estimated from transformed signals [cf. (20)] depend on expected values of the form $E[\hat{R}^{(T)}_{X_{Y_1}X_{Z_1}}(\hat{R}^{(T)}_{X_{Y_2}X_{Z_2}})^*]$, where $\hat{R}^{(T)}_{X_{Y_i}X_{Z_i}}$ are sample-time zero-lag auto- or cross-correlation estimates given by (9) (Y_1, Y_2, Z_1, and Z_2 can be either H or V to denote transformed signals from either polarimetric channel). This generic quantity can be expanded as

$$E[\hat{R}^{(T)}_{X_{Y_1}X_{Z_1}}(\hat{R}^{(T)}_{X_{Y_2}X_{Z_2}})^*] = \frac{1}{L^2}\sum_{l=0}^{L-1}\sum_{l'=0}^{L-1} E[X^*_{Y_1}(l)X_{Z_1}(l)X_{Y_2}(l')X^*_{Z_2}(l')], \tag{32}$$

where the sample-time dependence is purposely omitted because we are looking at transformations that only affect correlations along range time. For zero-mean complex Gaussian random variables, (32) can be simplified as (Reed 1962)

$$\begin{aligned} E[\hat{R}^{(T)}_{X_{Y_1}X_{Z_1}}(\hat{R}^{(T)}_{X_{Y_2}X_{Z_2}})^*] &= R_{X_{Y_1}X_{Z_1}}R_{X_{Z_2}X_{Y_2}} + \frac{1}{L^2}\sum_{l=0}^{L-1}\sum_{l'=0}^{L-1} R_{X_{Y_1}X_{Y_2}}(l'-l,0)R_{X_{Z_2}X_{Z_1}}(l-l',0) \\ &= R_{X_{Y_1}X_{Z_1}}R^*_{X_{Y_2}X_{Z_2}} + R_{X_{Y_1}X_{Y_2}}R^*_{X_{Z_1}X_{Z_2}}\frac{1}{L^2}\mathrm{tr}(\mathbf{C}_{X_{Y_1}X_{Y_2}}\mathbf{C}^{*\mathrm{T}}_{X_{Z_1}X_{Z_2}}). \end{aligned} \tag{33}$$

This result will be used next to express the variance of Z_{DR}, Φ_{DP}, and ρ_{HV} as functions of the H and V transformation matrices.

The differential reflectivity estimator for transformed oversampled data using simultaneous transmission and reception of horizontally and vertically polarized signals is given by

$$\hat{Z}_{\mathrm{DR}} = \frac{\hat{R}^{(T)}_{X_HX_H}}{\hat{R}^{(T)}_{X_VX_V}}. \tag{34}$$

Hence, for a normally biased Z_{DR} estimator, we need a set of transformations that leads to unbiased estimates of autocorrelations (see the appendix); that is, transformed signals are obtained as $\mathbf{x}_{H,V} = \mathbf{W}_{H,V}\,\mathbf{v}_{H,V}$, where $\mathbf{W}_{H,V}$ satisfies $\mathrm{tr}(\mathbf{W}^*_{H,V}\mathbf{C}_{V_{H,V}V_{H,V}}\mathbf{W}^{\mathrm{T}}_{H,V}) = L$. The variance of the estimator in (34) was given by Sachidananda and Zrnić (1985) and is adapted here for transformed oversampled signals as

$$\mathrm{Var}(\hat{Z}_{\mathrm{DR}}) = Z^2_{\mathrm{DR}}\left[\mathrm{Var}\left(\frac{\hat{R}^{(T)}_{X_HX_H}}{R^{(T)}_{X_HX_H}}\right) + \mathrm{Var}\left(\frac{\hat{R}^{(T)}_{X_VX_V}}{R^{(T)}_{X_VX_V}}\right) - 2\,\mathrm{Cov}\left(\frac{\hat{R}^{(T)}_{X_HX_H}}{R^{(T)}_{X_HX_H}}, \frac{\hat{R}^{(T)}_{X_VX_V}}{R^{(T)}_{X_VX_V}}\right)\right]. \tag{35}$$

Using (33) and after mathematical manipulation, this expression becomes

$$\mathrm{Var}(\hat{Z}_{\mathrm{DR}}) = \frac{Z_{\mathrm{DR}}^2}{L^2}[\mathrm{tr}(\mathbf{C}_{X_H X_H}^2) + \mathrm{tr}(\mathbf{C}_{X_V X_V}^2) - 2\rho_{HV}^2 \mathrm{tr}(\mathbf{C}_{X_H X_V}\mathbf{C}_{X_H X_V}^{*\mathrm{T}})], \tag{36}$$

where

$$\mathbf{C}_{X_H X_H} = \mathbf{W}_H^* \mathbf{C}_{V_H V_H} \mathbf{W}_H^{\mathrm{T}}, \tag{37}$$

$$\mathbf{C}_{X_V X_V} = \mathbf{W}_V^* \mathbf{C}_{V_V V_V} \mathbf{W}_V^{\mathrm{T}}, \quad \text{and} \tag{38}$$

$$\mathbf{C}_{X_H X_V} = \mathbf{W}_H^* \mathbf{C}_{V_H V_V} \mathbf{W}_V^{\mathrm{T}}. \tag{39}$$

The differential phase estimator is given by

$$\hat{\Phi}_{\mathrm{DP}} = \arg(\hat{R}_{\tilde{X}_H \tilde{X}_V}^{(T)}). \tag{40}$$

Hence, for a normally biased Φ_{DP} estimator, we need a set of transformations that lead to unbiased estimates of the cross correlation (see the appendix). In this case, transformed signals are obtained as $\tilde{X}_{H,V} = \tilde{\mathbf{W}}_{H,V}\mathbf{v}_{H,V}$, where $\tilde{\mathbf{W}}_{H,V}$ satisfy $\mathrm{tr}(\tilde{\mathbf{W}}_H^* \mathbf{C}_{V_H V_V} \tilde{\mathbf{W}}_V^{\mathrm{T}}) = L$. The variance of the estimator in (40) was given by Ryzhkov and Zrnić (1998) and is adapted here as

$$\mathrm{Var}(\hat{\Phi}_{\mathrm{DP}}) = \frac{1}{2}\mathrm{Re}\left\{E\left[\left|\frac{\hat{R}_{\tilde{X}_H \tilde{X}_V}^{(T)}}{R_{\tilde{X}_H \tilde{X}_V}^{(T)}}\right|^2\right] - E\left[\left(\frac{\hat{R}_{\tilde{X}_H \tilde{X}_V}^{(T)}}{R_{\tilde{X}_H \tilde{X}_V}^{(T)}}\right)^2\right]\right\}. \tag{41}$$

Using similar manipulations, this expression becomes

$$\mathrm{Var}(\hat{\Phi}_{\mathrm{DP}}) = \frac{1}{2L^2}\mathrm{Re}[\rho_{HV}^{-2}\mathrm{tr}(\mathbf{C}_{\tilde{X}_H \tilde{X}_H}\mathbf{C}_{\tilde{X}_V \tilde{X}_V}) - \mathrm{tr}(\mathbf{C}_{\tilde{X}_H \tilde{X}_V}^2)]. \tag{42}$$

The correlation matrices in this equation are similar to (37), (38), and (39), except that Xs and $\mathbf{W}$s carry a tilde.

Finally, the magnitude of the cross-correlation coefficient estimator is given by

$$\hat{\rho}_{HV} = \frac{|\hat{R}_{\tilde{X}_H \tilde{X}_V}^{(T)}|}{\sqrt{\hat{R}_{X_H X_H}^{(T)}\hat{R}_{X_V X_V}^{(T)}}}. \tag{43}$$

Unlike the previous two estimators, for a normally biased estimator of ρ_{HV}, we need two sets of transformations—one for unbiased autocorrelation estimates and another for unbiased cross-correlation estimates—that is, two sets of transformed signals are needed. As shown in (43), the tilde is again used to distinguish the set of transformed data used in the estimation of the cross correlation. The variance of ρ_{HV} estimates for simultaneous transmission and reception of horizontally and vertically polarized signals was given by Torlaschi and Gingras (2003) and is adapted here as

$$\begin{aligned}\mathrm{Var}(\hat{\rho}_{HV}) = \rho_{HV}^2\Bigg(&E\left\{\left[\mathrm{Re}\left(\frac{\hat{R}_{\tilde{X}_H \tilde{X}_V}^{(T)}}{R_{\tilde{X}_H \tilde{X}_V}^{(T)}} - 1\right)\right]^2\right\} - \mathrm{Cov}\left[\mathrm{Re}\left(\frac{\hat{R}_{\tilde{X}_H \tilde{X}_V}^{(T)}}{R_{\tilde{X}_H \tilde{X}_V}^{(T)}}\right), \frac{\hat{R}_{X_H X_H}^{(T)}}{R_{X_H X_H}^{(T)}}\right] - \mathrm{Cov}\left[\mathrm{Re}\left(\frac{\hat{R}_{\tilde{X}_H \tilde{X}_V}^{(T)}}{R_{\tilde{X}_H \tilde{X}_V}^{(T)}}\right), \frac{\hat{R}_{X_V X_V}^{(T)}}{R_{X_V X_V}^{(T)}}\right]\\ &+ \frac{1}{4}\mathrm{Var}\left(\frac{\hat{R}_{X_H X_H}^{(T)}}{R_{X_H X_H}^{(T)}}\right) + \frac{1}{4}\mathrm{Var}\left(\frac{\hat{R}_{X_V X_V}^{(T)}}{R_{X_V X_V}^{(T)}}\right) + \frac{1}{2}\mathrm{Cov}\left(\frac{\hat{R}_{X_H X_H}^{(T)}}{R_{X_H X_H}^{(T)}}, \frac{\hat{R}_{X_V X_V}^{(T)}}{R_{X_V X_V}^{(T)}}\right)\Bigg).\end{aligned} \tag{44}$$

In terms of normalized auto- and cross-correlation matrices, this expression becomes

$$\begin{aligned}\mathrm{Var}(\hat{\rho}_{HV}) = \frac{\rho_{HV}^2}{L^2}\Bigg\{&\mathrm{Re}\left[\frac{1}{2\rho_{HV}^2}\mathrm{tr}(\mathbf{C}_{\tilde{X}_H \tilde{X}_H}\mathbf{C}_{\tilde{X}_V \tilde{X}_V}) + \frac{1}{2}\mathrm{tr}(\mathbf{C}_{\tilde{X}_H \tilde{X}_V}^2)\right]\\ &- \mathrm{Re}[\mathrm{tr}(\mathbf{C}_{X_H \tilde{X}_V}\mathbf{C}_{\tilde{X}_H X_H}) + \mathrm{tr}(\mathbf{C}_{X_V \tilde{X}_V}\mathbf{C}_{\tilde{X}_H X_V})]\\ &+ \frac{1}{4}\mathrm{tr}(\mathbf{C}_{X_H X_H}^2) + \frac{1}{4}\mathrm{tr}(\mathbf{C}_{X_V X_V}^2) + \frac{\rho_{HV}^2}{2}\mathrm{tr}(\mathbf{C}_{X_H X_V}\mathbf{C}_{X_H X_V}^{*T})\Bigg\},\end{aligned} \tag{45}$$

where correlation matrices are defined as in (37)–(39), except that tilde and nontilde matrices appear at the same time in some cases (e.g., $\mathbf{C}_{\tilde{X}_H X_V} = \tilde{\mathbf{W}}_H^* \mathbf{C}_{V_H V_V} \mathbf{W}_V^{\mathrm{T}}$).

It is important to recall that the use of a single optimum transformation matrix applies only if polarimetric channels are perfectly matched. If this is not the case, each polarimetric variable requires its own transformation matrix set for optimum estimates (i.e., two transformation matrices for Z_{DR}, another two for Φ_{DP}, and yet another four for ρ_{HV}). These variable-specific transformation sets are obtained by solving (31) with the functions in (36), (42), and (45). As in the case of matched channels, these transformation sets can be precomputed to reduce the computational complexity of the optimum unbiased estimators. However, because the transformations arising from the constrained minimization of (36), (42), and (45) depend on ρ_{HV}, a transformation set should be precomputed for every single value of ρ_{HV} between 0 and 1—a daunting task. In practice, it suffices to precompute transformation sets for a finite set of ρ_{HV} values (e.g., 20 sets would cover the range of ρ_{HV} values with a resolution of 0.05) and select (in real time) the set that best matches the situation at hand to get an ''almost optimum'' estimator performance. Additionally, ρ_{HV} is not known a priori but is one of the variables to be estimated from oversampled data. To handle this apparent paradox, an initial estimate of ρ_{HV} could be obtained from non-oversampled (or decimated) data to select the proper transformation set using a lookup table. Even with precomputed sets of transformations, a real-time implementation of optimum unbiased transformation (OUTB) estimators on polarimetric radars with mismatched channels would be computationally more expensive than biased whitening (WTB) estimators on systems with matched channels; that is, oversampled time series data for the H and V channels have to be transformed 4 times each, instead of just once as in (14). Also, estimates of auto- and cross correlations cannot be ''shared'' among the three polarimetric variable estimators. Each variable requires its own correlation estimators. All in all, a real-time implementation of OUTB estimators should be feasible with modern digital signal processing technology.

6. Simulation results

Simulated time series data are used to study the effects of channel mismatches on the estimation of polarimetric variables from range-oversampled data. In particular, the results provide validation of the ''unbiasing'' scaling presented in section 4 and show the performance of the optimum transformations derived in section 5. Signals are simulated as described by Torres and Zrnić (2003a,b) using varying degrees of mismatch between the modified pulses of the H and V channels. The modified pulse for the H channel serves as a reference and is fixed with rectangular amplitude and zero phase as

$$p_H(l) = \begin{cases} L^{-1/2} & 0 \le l < L \\ 0 & \text{otherwise} \end{cases}. \qquad (46)$$

Although this pulse shape may be unrealistic for practical systems, the results that follow are equally applicable to any specific pulse shape. Recall that it is the mismatch between the H and V channels that leads to biased correlation estimators. The modified pulse for the V channel is varied to obtain different degrees of mismatch as

$$p_V(l) = \begin{cases} [\alpha_0 + \alpha_1 p_\alpha(l)] \exp\{j[\beta_0 + \beta_1 p_\beta(l)]\} p_H(l) & 0 \le l < L \\ 0 & \text{otherwise} \end{cases}, \qquad (47)$$

where unitless factors α_0 and β_0 control constant amplitude and phase mismatches and unitless factors α_1 and β_1 control time-varying amplitude and phase mismatches (p_α and p_β are unitless functions of range time l). Note that perfect matching is obtained if $\alpha_0 = 1$ and $\alpha_1 = \beta_0 = \beta_1 = 0$. For this simulation study, we consider two types of functions for p_α and p_β: a linearly increasing function

$$p_{\text{ramp}}(l) = \frac{l}{L-1}, \quad l = 0, 1, \ldots, L-1 \qquad (48)$$

and a triangular function[2]

$$p_{\text{triang}}(l) = 1 - \frac{2}{L-1}\left| l - \frac{L-1}{2} \right|, \quad l = 0, 1, \ldots, L-1. \qquad (49)$$

[2] This formula is for odd L.

The pattern for these mismatches is based on Choudhury and Chandrasekar's (2007) work on the CSU–CHILL radar and on our observations from the National Severe Storms Laboratory Next Generation Weather Radar (NEXRAD) polarimetric prototype (Ivić et al. 2003b). An amplitude mismatch could be due to miscalibrated pulse-forming networks or different overall gains in each channel. A phase mismatch might be attributed to a known effect with klystron amplifiers. It has been observed that these devices exhibit an AM-to-PM conversion whereby voltage variations in the transmitted pulse envelope are converted into phase variations of the carrier (Ivić et al. 2003b).

Polarimetric range-oversampled weather-like data are simulated for varying degrees of channel mismatch to illustrate the performance of the transformations developed in the previous sections. For each mismatch case, 1000 realizations of time series data are generated with the following parameters: the oversampling factor

TABLE 1. Summary of abnormally biased range-oversampling polarimetric variable estimators for different amplitude and phase mismatch cases for oversampling and averaging (OAB), biased whitening (WTB), unbiased whitening (UWTB), and optimum unbiased transformation (OUTB). The amplitude and phase mismatch parameters are those in (47), and ramp and triang. correspond to Eqs. (48) and (49), respectively.

	Amplitude mismatch			Phase mismatch			Abnormally biased estimators		
Case	α_0	α_1	p_α	β_0	β_1	p_β	OAB	WTB	UWTB/OUTB
1	1.0	0	ramp	0	0	ramp	—	—	—
2	0.8	0	ramp	0	0	ramp	Z_{DR}	Z_{DR}	—
3	1.0	0	ramp	$\pi/6$	0	ramp	Φ_{DP}	Φ_{DP}	—
4	0.8	0	ramp	$\pi/6$	0	ramp	Z_{DR}, Φ_{DP}	Z_{DR}, Φ_{DP}	—
5	0.8	0.2	ramp	0	0	ramp	Z_{DR}, ρ_{HV}	Z_{DR}, ρ_{HV}	—
6	1.0	0	ramp	0	$\pi/6$	ramp	Φ_{DP}, ρ_{HV}	Φ_{DP}, ρ_{HV}	—
7	0.8	0.2	ramp	0	$\pi/6$	ramp	Z_{DR}, Φ_{DP}, ρ_{HV}	Z_{DR}, Φ_{DP}, ρ_{HV}	—
8	0.8	0.2	triang.	0	$\pi/6$	ramp	Z_{DR}, Φ_{DP}, ρ_{HV}	Z_{DR}, Φ_{DP}, ρ_{HV}	—

$L = 5$, the number of samples in the dwell time $M = 64$, the signal-to-noise ratio is very large, and the Nyquist velocity is 32 m s^{-1}. The spectral moments are $S_H = 0$ dB, $v = 0$ m s^{-1}, and $\sigma_v = 4$ m s^{-1}; the polarimetric variables are $Z_{DR} = 1$ dB, $\Phi_{DP} = 30°$, and $\rho_{HV} = 0.985$. Range-oversampled data are processed using four matrix transformation sets:

1) oversampling and averaging (OAB), in which oversampled signals are not transformed; that is, $\mathbf{W}_{H,V} = \tilde{\mathbf{W}}_{H,V} = \mathbf{I}$;
2) WTB, in which the same whitening transformation (the one for the H channel) is used disregarding channel mismatch (section 4b); that is, $\mathbf{W}_{H,V} = \tilde{\mathbf{W}}_{H,V} = \mathbf{H}^{-1}$, where $\mathbf{C}_{V_H V_H} = \mathbf{H}^*\mathbf{H}^T$;
3) UWTB, in which the proper scaling factors are applied (section 4d), $\mathbf{W}_{H,V}$ are the whitening transformations for each channel, and $\tilde{\mathbf{W}}_{H,V}$ diagonalize the normalized range cross-correlation matrix $\mathbf{C}_{V_H V_V}$; and
4) OUTB, in which three sets of transformations are obtained by solving the constrained minimization problems presented in section 5. These problems are solved using the sequential quadratic programming (SQP) method implemented in MATLAB function "fmincon." SQP methods are best suited for solving problems with significant nonlinearities in the constraints (Nocedal and Wright 2006), which is the case in our formulation.

Table 1 shows a list of abnormally biased polarimetric variable estimators on range-oversampled data for different channel mismatch cases. Case 1 is the ideal case of matched channels in which all the estimators are unbiased for all the transformations under consideration. Cases 2, 3, and 4 correspond to a constant mismatch in amplitude, phase, and both, respectively. Unless mismatches are properly accounted for through unbiasing factors (UWTB and OUTB), amplitude mismatches lead to an abnormal bias in Z_{DR} and phase mismatches lead to an abnormal bias in Φ_{DP}. Note that these constant mismatches can be corrected via Z_{DR} and Φ_{DP} calibration constants; thus, biases in these polarimetric variables are typically of little concern. Conversely, biases in ρ_{HV} are more difficult to correct through calibration; nevertheless, in these cases ρ_{HV} estimates remain normally biased. Cases 5, 6, and 7 correspond to time-varying mismatches of the ramp form [cf. (48)] in amplitude, phase, and both, respectively. Finally, case 8 is similar to case 7 but uses (49) for the time-varying amplitude mismatch. Analogously to the previous cases, amplitude (phase) mismatches lead to abnormally biased OAB and WTB estimates of Z_{DR} (Φ_{DP}), which can be removed through calibration. However, in all these cases, OAB and WTB estimates of ρ_{HV} are abnormally biased and removal of this bias is not straightforward using calibration techniques.

Next, varying degrees of time-dependent mismatches are evaluated. Figures 1 and 2 show the bias and standard deviation of the polarimetric variables corresponding to varying degrees of phase and amplitude mismatch, respectively. These figures show that although WTB has lower standard deviation than OAB (except for ρ_{HV} estimates with phase mismatches with β_1 larger than about $\pi/9$), it produces abnormally biased estimates of polarimetric variables as the degree of mismatch between the H and V channels increases (note that whereas significant Z_{DR} biases correspond to amplitude mismatches and Φ_{DP} biases only occur for phase mismatches, ρ_{HV} biases are of practical concern in both types of mismatches). On the other hand, UWTB has almost the same or better variance reduction as WTB but produces normally biased estimates irrespective of the degree of mismatch between the H and V channels. Notice, however, that the standard deviations of estimates with UWTB (and WTB) increase relative to those of OAB as the degree of mismatch between the H and V channels increases. Indeed, for large degrees of

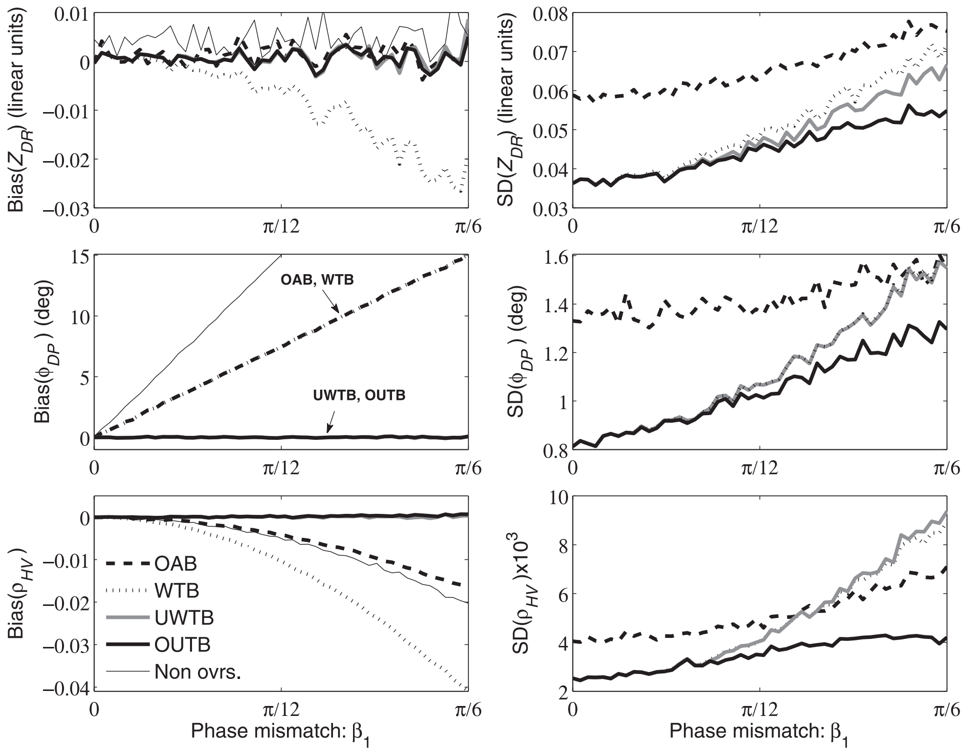

FIG. 1. (left) Bias and (right) standard deviation of polarimetric variable estimates for oversampled data processed using OAB, WTB, UWTB, and OUTB. In addition, the biases of estimates for standard processing (non-oversampled data) without calibration are included as a reference. The H and V channels exhibit a linearly increasing phase mismatch with zero initial phase ($\beta_0 = 0$) and maximum departure (β_1) varying from 0 to $\pi/6$.

mismatch, UWTB performs worse than OAB (in this specific case, this is mainly observed for ρ_{HV} with phase mismatches with β_1 larger than about $\pi/9$). Conversely, in all cases, OUTB achieves maximum variance reduction of polarimetric variable estimates with minimum bias as predicted theoretically.

7. Conclusions

This paper demonstrates that, by properly accounting for the amplitude and/or phase differences in the transmission channels (i.e., by properly scaling the transformation matrices), it is always possible to obtain unbiased polarimetric variable estimates from range-oversampled signals. Nevertheless, as shown by the simulations, the variance of these estimators degrades as the degree of mismatch between the horizontally and the vertically polarized transmitted pulses increases. Still, by solving a constrained minimization problem, it is possible to find transformation structures that result in unbiased auto- and cross-correlation estimates and at the same time achieve maximum variance reduction.

Although it was shown that polarimetric channel mismatches can be properly accounted for, the implementation of optimum unbiased estimators on range-oversampled signals comes at a price because of the additional complexity and computational requirements. However, depending on the maximum acceptable polarimetric variable biases, the conventional use of whitening transformations may still be possible on systems with small channel mismatches; that is, assuming that Z_{DR} and Φ_{DP} biases can be effectively removed through calibration procedures, the maximum acceptable ρ_{HV} bias can be used to determine the worst channel mismatch conditions for which a whitening transformation is still viable (e.g., for the cases depicted in Figs. 1, 2, a maximum acceptable ρ_{HV} bias of 0.01 would result in whitening being applicable for amplitude mismatches less than about 0.175 or phase mismatches less than about $\pi/12$). However, on systems with significant channel mismatches, one might be forced to implement the more-involved optimum unbiased estimators. Fortunately, on polarization-diverse single-transmitter radar systems, such as the planned upgrades of the NEXRAD network (Doviak et al. 2000), significant channel mismatches are not likely to occur.

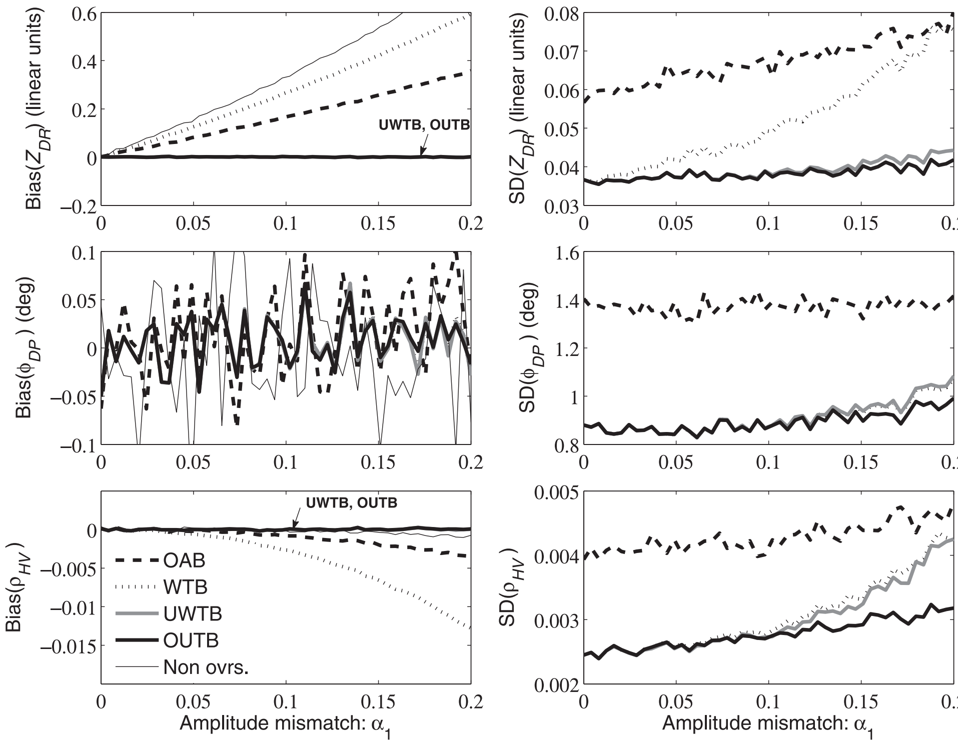

FIG. 2. Bias (left) and standard deviation (right) of polarimetric variable estimates for oversampled data processed using: (1) oversampling and averaging (OAB), (2) biased whitening (WTB), (3) unbiased whitening (UWTB), and (4) optimum unbiased transformation (OUTB). In addition, the biases of estimates for standard processing (non-oversampled data) without calibration are included as a reference. The H and V channels exhibit a progressively increasing amplitude mismatch with α_1 varying from 0 to 0.2 and $\alpha_0 + \alpha_1 = 1$.

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APPENDIX

Bias of Polarimetric Variable Estimators

Throughout this paper, it is assumed that unbiased correlation estimates are required to preserve the integrity of the polarimetric variables; this condition motivates the focus on correlation estimates in section 2 and is used as a constraint to the minimization problem in section 5. However, polarimetric variable estimators are nonlinear functions of auto- and cross-correlation estimates; thus, unbiased correlation estimates do not necessarily lead to unbiased Z_{DR}, ρ_{HV}, and Φ_{DP} estimates. In fact, polarimetric variable estimators are inherently biased (or normally biased), as shown by Melnikov and Zrnić (2004). Nevertheless, a condition to ensure that biases of polarimetric variables obtained from range-oversampled signals are not any larger than those obtained with standard processing (no oversampling) is that correlation estimators based on range-oversampled signals be unbiased. This is shown next.

Consider a generic estimator $\hat{y}$ derived as a function of n primary estimators: $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$; that is, $\hat{y} = g(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$, where g is any function. According to Papoulis (1984), if g is sufficiently smooth near the point $(E[\hat{x}_1], E[\hat{x}_2], \ldots, E[\hat{x}_n])$, the expected value of the derived estimator can be approximated as

$$E[\hat{y}] \approx g + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 g}{\partial \hat{x}_i \partial \hat{x}_j}\mathrm{Cov}[\hat{x}_i, \hat{x}_j], \qquad \text{(A1)}$$

where g and its derivatives are evaluated at $(E[\hat{x}_1], E[\hat{x}_2], \ldots, E[\hat{x}_n])$ and Cov is the covariance operator. Note that if g is a linear function, the terms from the double sum vanish because the second derivatives of a linear function are zero.

Based on (A1), the bias of the derived estimator given by $\mathrm{Bias}[\hat{y}] = E[\hat{y}] - y$ can be approximated as

$$\mathrm{Bias}[\hat{y}] \approx g(E[\hat{x}_1], \ldots, E[\hat{x}_n]) - g(x_1, \ldots, x_n) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 g}{\partial\hat{x}_i\partial\hat{x}_j}\mathrm{Cov}[\hat{x}_i, \hat{x}_j]. \tag{A2}$$

Hence, the bias of the derived estimator depends not only on the biases of the primary estimators but also on their covariances. Note that if the primary estimators are unbiased [i.e., $E[\hat{x}_i] = x_i\,(i = 1, \ldots, n)$], the bias of the derived estimator depends only on the covariance terms; that is,

$$\mathrm{Bias}[\hat{y}] \approx \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 g}{\partial\hat{x}_i\partial\hat{x}_j}\mathrm{Cov}[\hat{x}_i, \hat{x}_j] \tag{A3}$$

because

$$g(E[\hat{x}_1], \ldots, E[\hat{x}_n]) = g(x_1, \ldots, x_n), \tag{A4}$$

and the magnitude of the bias is bounded as

$$|\mathrm{Bias}[\hat{y}]| \le \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{\partial^2 g}{\partial\hat{x}_i\partial\hat{x}_j}\right|\{\mathrm{Var}[\hat{x}_i]\mathrm{Var}[\hat{x}_j]\}^{1/2}, \tag{A5}$$

where we used the fact that $|\mathrm{Cov}[\hat{x}_i,\hat{x}_j]| \le \sqrt{\mathrm{Var}[\hat{x}_i]\mathrm{Var}[\hat{x}_j]}$.

Polarimetric variable estimators are derived from correlation estimators (i.e., the primary estimators) as $\hat{Z}_{\mathrm{DR}} = g_1(\hat{R}^{(T)}_{X_H X_H}, \hat{R}^{(T)}_{X_V X_V})$, $\hat{\Phi}_{\mathrm{DP}} = g_2(\hat{R}^{(T)}_{X_H X_V})$, and $\hat{\rho}_{HV} = g_3(\hat{R}^{(T)}_{X_H X_H}, \hat{R}^{(T)}_{X_V X_V}, \hat{R}^{(T)}_{X_H X_V})$, where g_1, g_2, and g_3 are nonlinear complex functions. For these estimators, based on transformed oversampled signals, the variances in (A5) can be computed using (33) as

$$\mathrm{Var}[\hat{R}^{(T)}_{X_Y X_Z}] = R_{X_Y X_Y} R^*_{X_Z X_Z} \frac{1}{L^2}\mathrm{tr}(\mathbf{C}_{X_Y X_Y}\mathbf{C}_{X_Z X_Z}), \tag{A6}$$

where subscripts Y and Z can be either H or V.

Next, let us compare the biases of estimators based on standard processing with no oversampling to those based on transformed oversampled signals. The magnitude of the biases can be estimated using (A5) for both classes of estimators, where the only difference between these two types of estimators is in the factors of the form $\mathrm{tr}(\mathbf{C}_{X_Y X_Y}\mathbf{C}_{X_Z X_Z})/L^2$ that originate from using (A6) in (A5). On one hand, for estimators based on standard processing, these factors are always equal to 1 (because $L = 1$ and the normalized range correlation matrices reduce to scalars; i.e., $\mathbf{C}_{X_Y X_Y} = \mathbf{C}_{X_Z X_Z} = 1$). On the other hand, for estimators based on transformed oversampled signals, these factors are at most 1; that is,

$$\frac{1}{L^2}\mathrm{tr}(\mathbf{C}_{X_Y X_Y}\mathbf{C}_{X_Z X_Z}) \le \frac{1}{L^2}\mathrm{tr}(\mathbf{C}_{X_Y X_Y})\mathrm{tr}(\mathbf{C}_{X_Z X_Z}) = 1, \tag{A7}$$

which follows because the matrices inside the trace are positive semidefinite (they are autocorrelation matrices) and are constrained for unbiased correlation estimators as in (31).

In summary, provided that correlation estimates are unbiased, the biases of polarimetric variable estimators based on transformed oversampled signals are never larger than those of estimators based on standard processing.

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